EIGENTRIADS

A MUSICAL OFFERING

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joint work with
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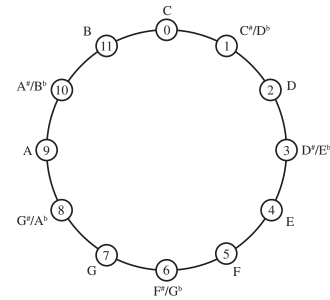
University of Mary Washington
where great minds get to work

The musical clock



What is a triad?

- By a *triad*, or *chord*, we mean three pitches played simultaneously
- For instance,
  - a C-Major triad consists of the pitches C, E and G.
  - an a-minor triad consists of the pitches A, C, and E.
- We represent triads as vectors over Z_{12}

C-Major = < 0, 4, 7 >
a-minor = < 4, 0, 9 >

Operations on triads

- The 19th century music theorist Hugo Riemann defined these operations

P – parallel,
L – leading tone exchange, and
R – relative

Examples:
P(C-Major) = c-minor
L(C-Major) = e-minor
R(C-Major) = a-minor

Mathematical descriptions of chord progressions

C-Major → a-minor → F-Major → G-Major

C → R(C) → L(R(C)) → T_2(L(R(C)))

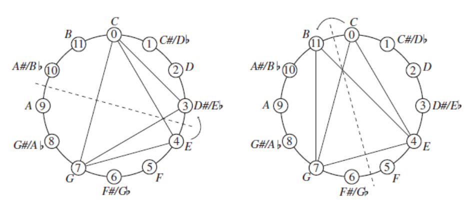
Stand by Me

Every Breath you Take

Those Magic Changes

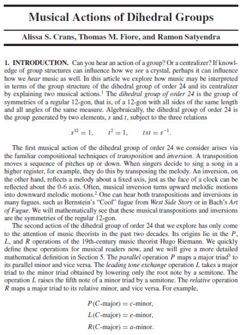
Eternal Flame

Dihedral actions?



PC = c   *LC = e*

Musical actions of dihedral groups



Musical Actions of Dihedral Groups

Alissa S. Crans, Thomas M. Fiore, and Ramon Satyendra

1. INTRODUCTION. Can you hear an action of a group? Or a centralizer? If knowledge of group structures can influence how we *see* a crystal, perhaps it can influence how we *hear* music as well. In this article we explore how music may be interpreted in terms of the group structure of the dihedral group of order 24 and its centralizer by explaining two musical actions.[1] The *dihedral group of order 24* is the group of symmetries of a regular 12-gon, that is, of a 12-gon with all sides of the same length and all angles of the same measure. Algebraically, the dihedral group of order 24 is the group generated by two elements, s and t, subject to the three relations

$$s^{12} = 1, \qquad t^2 = 1, \qquad tst = s^{-1}.$$

The first musical action of the dihedral group of order 24 we consider arises via the familiar compositional techniques of *transposition* and *inversion*. A transposition moves a sequence of pitches up or down. When singers decide to sing a song in a higher register, for example, they do this by transposing the melody. An inversion, on the other hand, reflects a melody about a fixed axis, just as the face of a clock can be reflected about the 0-6 axis. Often, musical inversion turns upward melodic motions into downward melodic motions.[2] One can hear both transpositions and inversions in many fugues, such as Bernstein's "Cool" fugue from *West Side Story* or in Bach's *Art of Fugue*. We will mathematically see that these musical transpositions and inversions are the symmetries of the regular 12-gon.

The second action of the dihedral group of order 24 that we explore has only come to the attention of music theorists in the past two decades. Its origins lie in the P, L, and R operations of the 19th-century music theorist Hugo Riemann. We quickly define these operations for musical readers now, and we will give a more detailed mathematical definition in Section 5. The *parallel* operation P maps a major triad[3] to its parallel minor and vice versa. The *leading tone exchange* operation L takes a major triad to the minor triad obtained by lowering only the root note by a semitone. The operation L raises the fifth note of a minor triad by a semitone. The *relative* operation R maps a major triad to its relative minor, and vice versa. For example,

$$P(C\text{-major}) = c\text{-minor},$$
$$L(C\text{-major}) = e\text{-minor},$$
$$R(C\text{-major}) = a\text{-minor}.$$

- Winner of the Hasse Award
- **Year of Award:** 2011
- *The American Mathematical Monthly*, vol. 116, no. 6, June 2009, pp. 479-495.

Eigenvectors…ur…eigen*triads*?

- We can consider musical triads that act as eigenvectors under one of these operators.
- We want to know what triad can satisfy:

$$P(v) = kv$$

where k is a "scalar"

Triads in terms of 'thirds'

Major and minor thirds

- A major third is 4 half steps
- A minor third is 3 half steps

Major and minor triads

- A Major triad is a major third followed by a minor third
- A minor triad is a minor third followed by a major third

Eigentriads

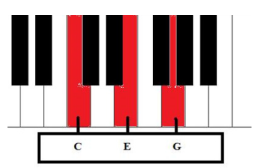
- Major and minor triads never show up as eigentriads
- Eigentriads need to have more "evenly spaced" notes
- So, what kind of triad consists of a pair of either major thirds or minor thirds?

Diminished and Augmented chords

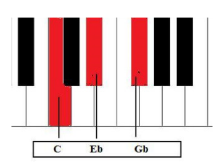
- A diminished chord is a pair of stacked minor thirds
- An augmented chord is a pair of stacked major thirds

Diminished chords

Major – 4 + 3



Diminished – 3 + 3



Michelle, by the Beatles

D Gm
Michelle, ma belle
C Ddim A
these are words that go together well,
Ddim A
my Michelle

All Star, by Smash Mouth

Simple version:
E A B A
Hey now, you're an all-star, get your game on, go play

With diminished:
E A A#dim A
Hey now, you're an all-star, get your game on, go play

As a transition chord

- Diminished chords also act as a nice transition between a whole step
- Rather than D – Em, use D – D#dim – Em

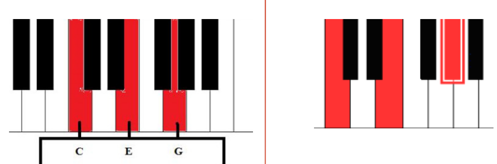
Example in Jingle Bell Rock:

D D#dim Em
jingle around the clock...

Augmented chords

Major – 4 + 3

Augmented – 4 + 4



From Me to You, by the Beatles

Gm C7
I got arms that long to hold you
F
and keep you by my side.
D7
I got lips that long to kiss you
G Gaug
And keep you satisfied

Possible musical meaning

- A possible musical reason for the augmented and diminished chords being eigentriads is that they do not change form, and therefore cannot stand alone melodically.
- The Major and minor triads can and do change parity under these three fundamental transformations: they can move and change into something else. But the augmented and diminished triads are stuck.

Other Eigentriads?

- P, L and R have many eigentriads, other than those we have already mentioned. However, most have no meaning musically (at least in Western music).
- For example, <7, 5, 6> is an eigenvector under R with eigenvalue 11.
- These collections of pitches could very well play a role in the music of other cultures.

Thank you

- Thanks to Salisbury University for hosting this event
- Thanks to my collaborators, Bud Brown and Alissa Crans
- Thanks for your attention