Applications of pencils of conics

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University of Mary Washington
(joint work with Jeremy Dover, Kelly Scott, and Kenny Wantz)

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Introduction to conics

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2-dimensional subspaces $\leftrightarrow$ lines

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$$y^2 = xz.$$

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So, a non-degenerate conic consists of the points whose homogeneous coordinates are

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or

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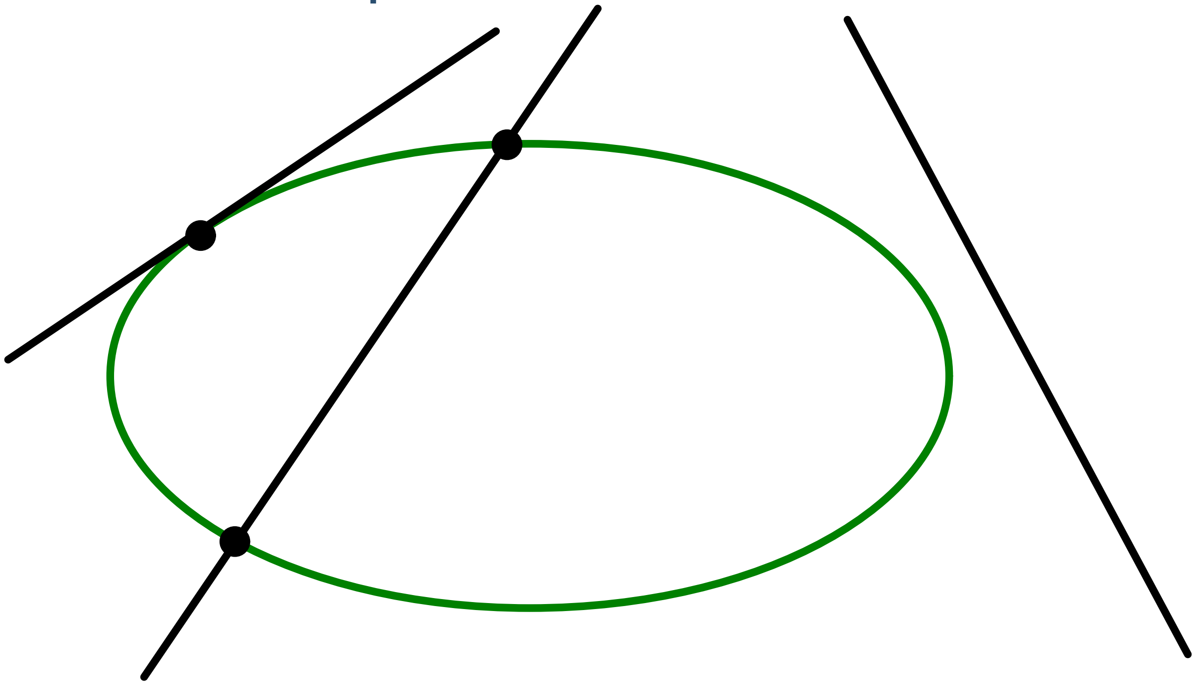
Every non-degenerate conic is an *arc*, a set of points, no three collinear. A famous theorem due to Segre says that when q is odd, every set of $q + 1$ points, no three collinear, is actually a conic.

synthetic properties

Since conics form arcs, no line can meet a conic in more than 2 points.



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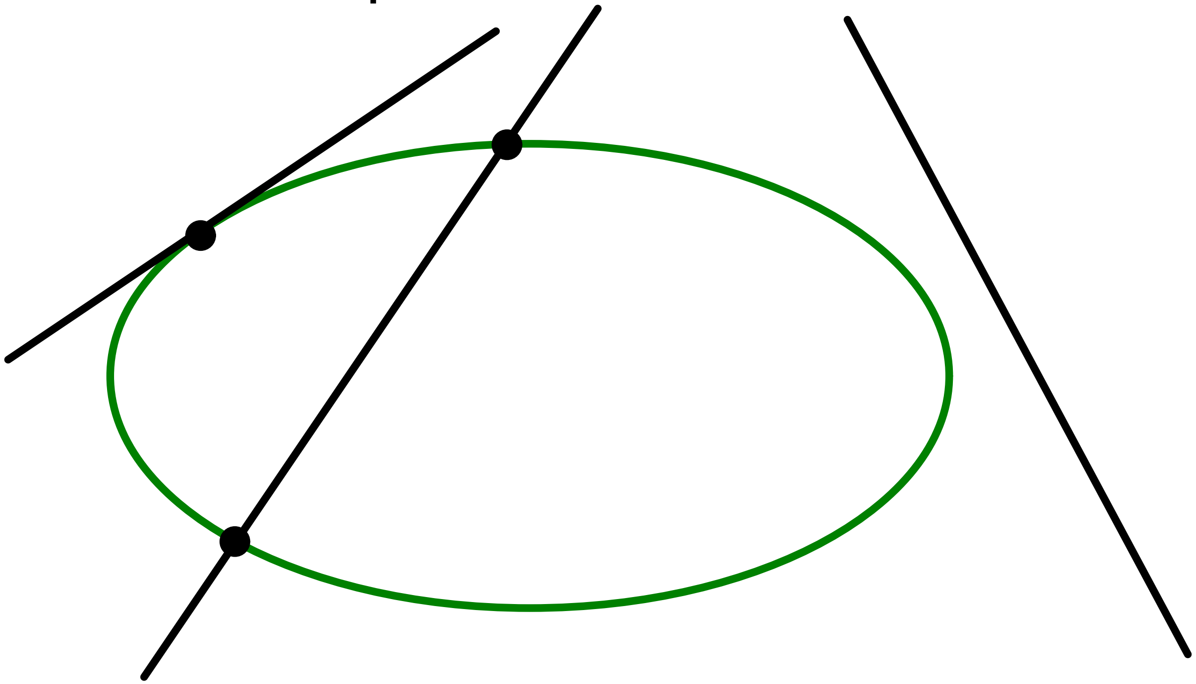

We have secant lines, tangent lines, and skew lines.

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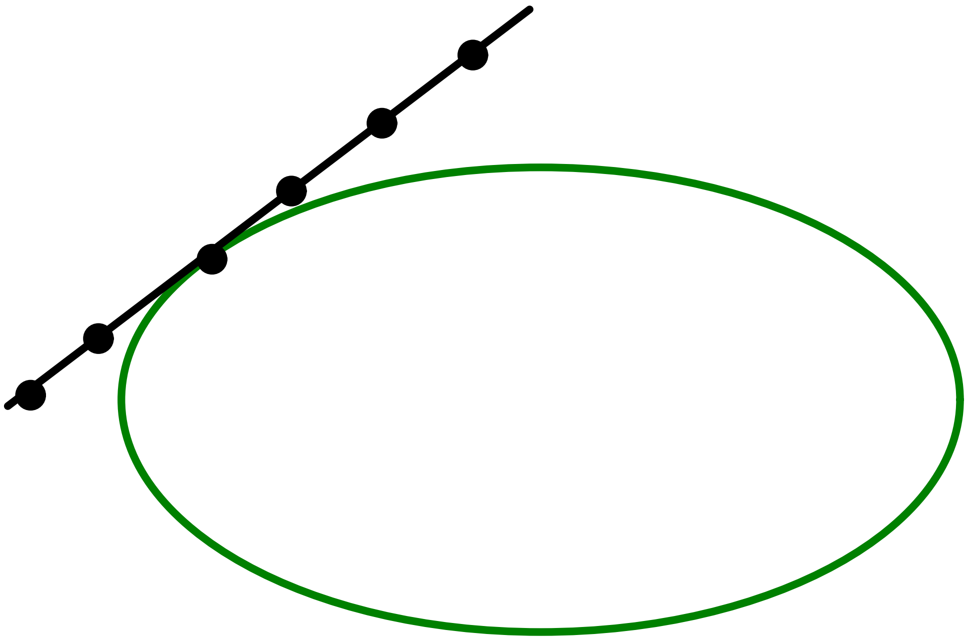

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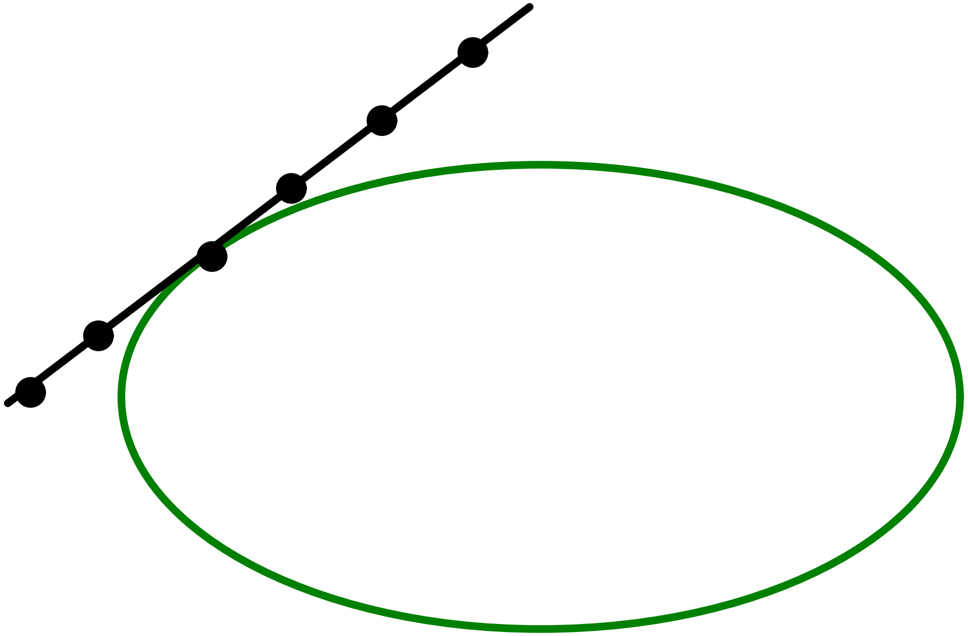

The *exterior* points are those that lie on tangent lines. Counting shows that about half the points off the conic are exterior. The other points are called *interior*.

pencils of conics

Let F and G be distinct quadratic forms. Then we can define the collection of quadratic forms $F + \lambda G$, $\lambda \in GF(q)$, to be the algebraic *pencil of conics* defined by F and G.

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There is a complete list in Hirschfeld's book – there are 20.

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Each involves a non-trivial construction of some object in the projective plane.

Semiovals

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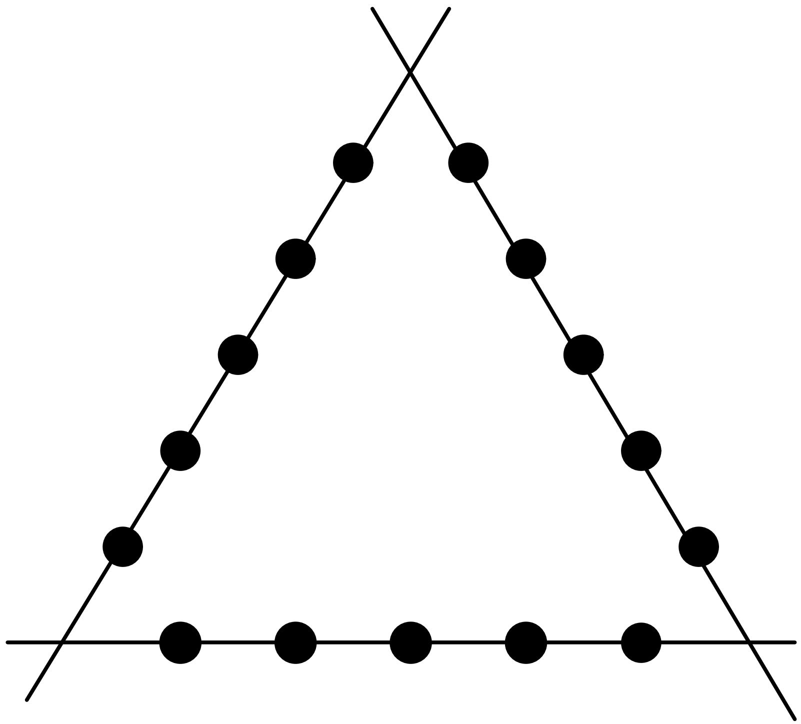
unique tangent lines

vertex-less triangle

Classic example: *the vertex-less triangle*



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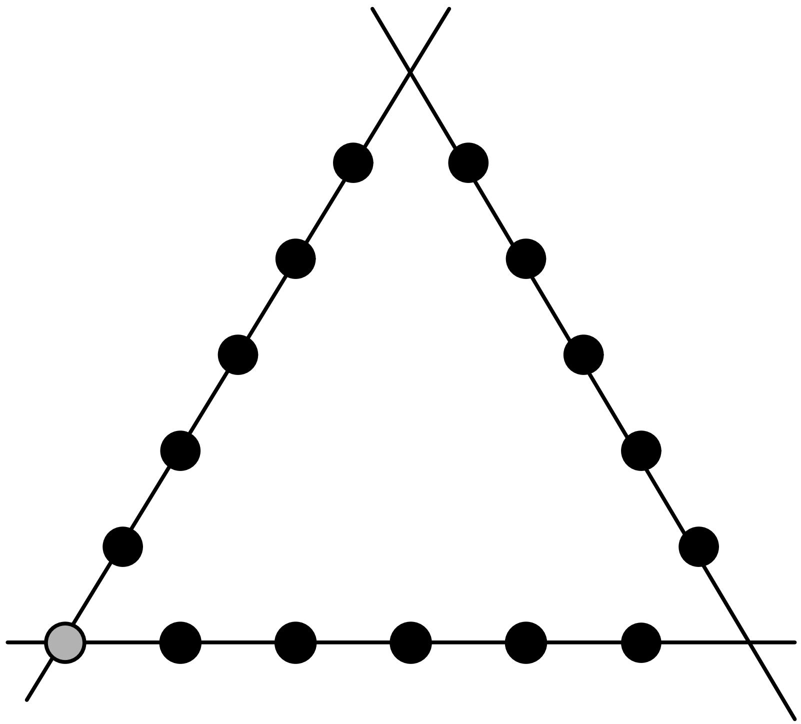

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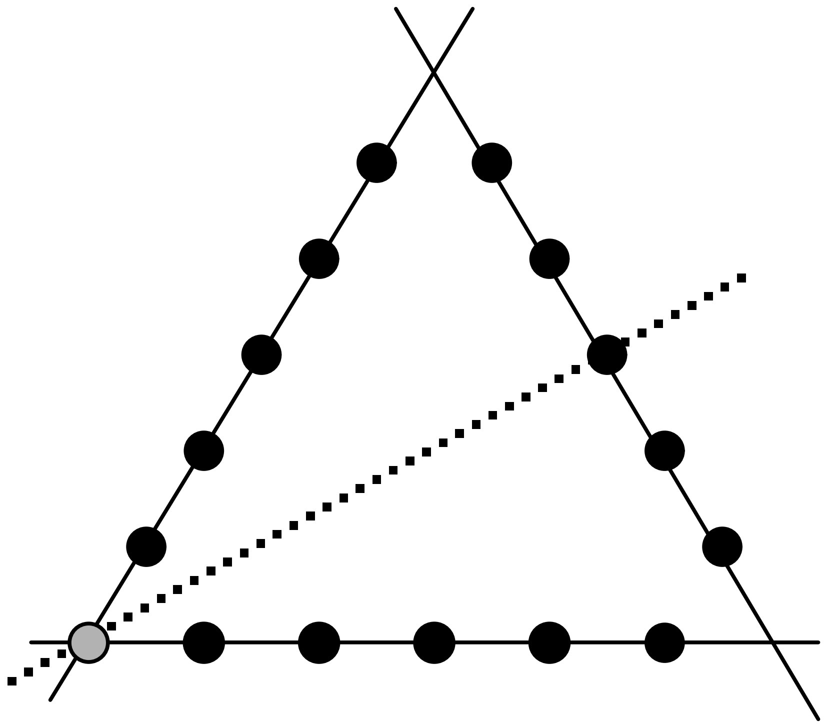

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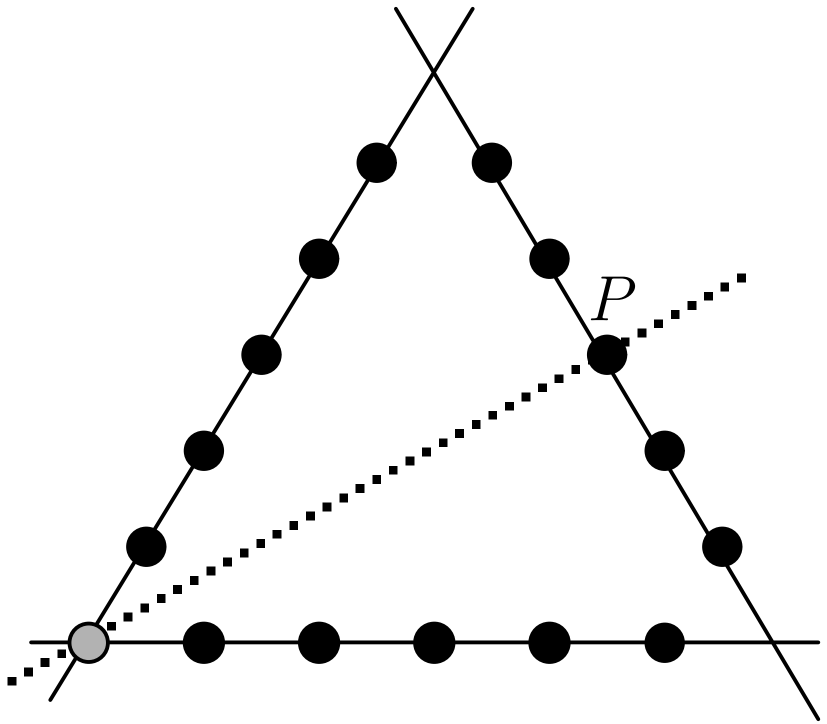

extremal in two ways

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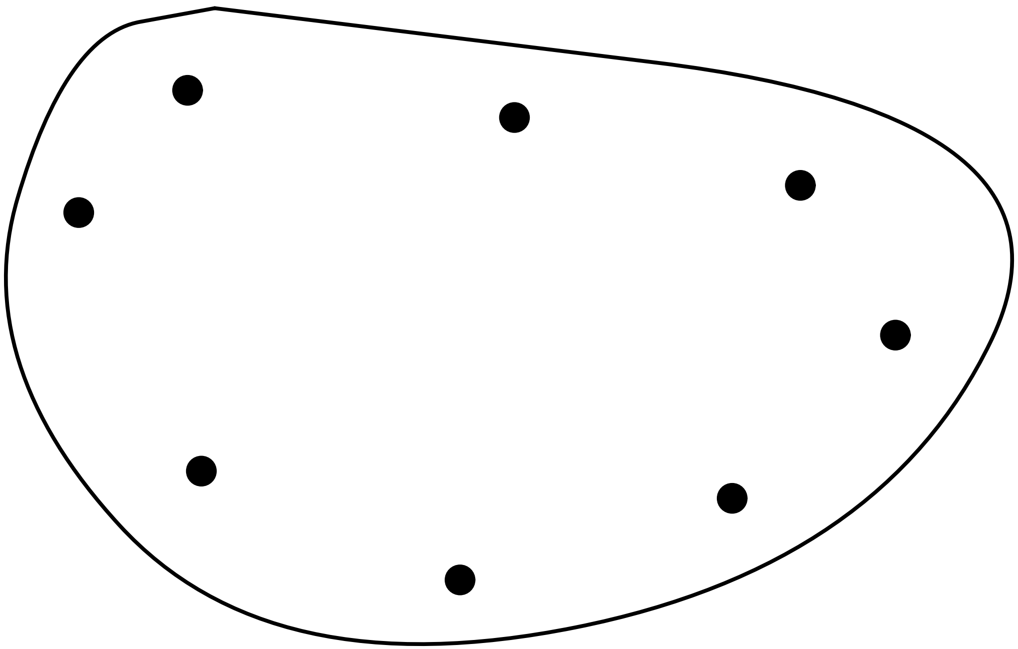
First, you can't remove any point and still have a blocking semioval.



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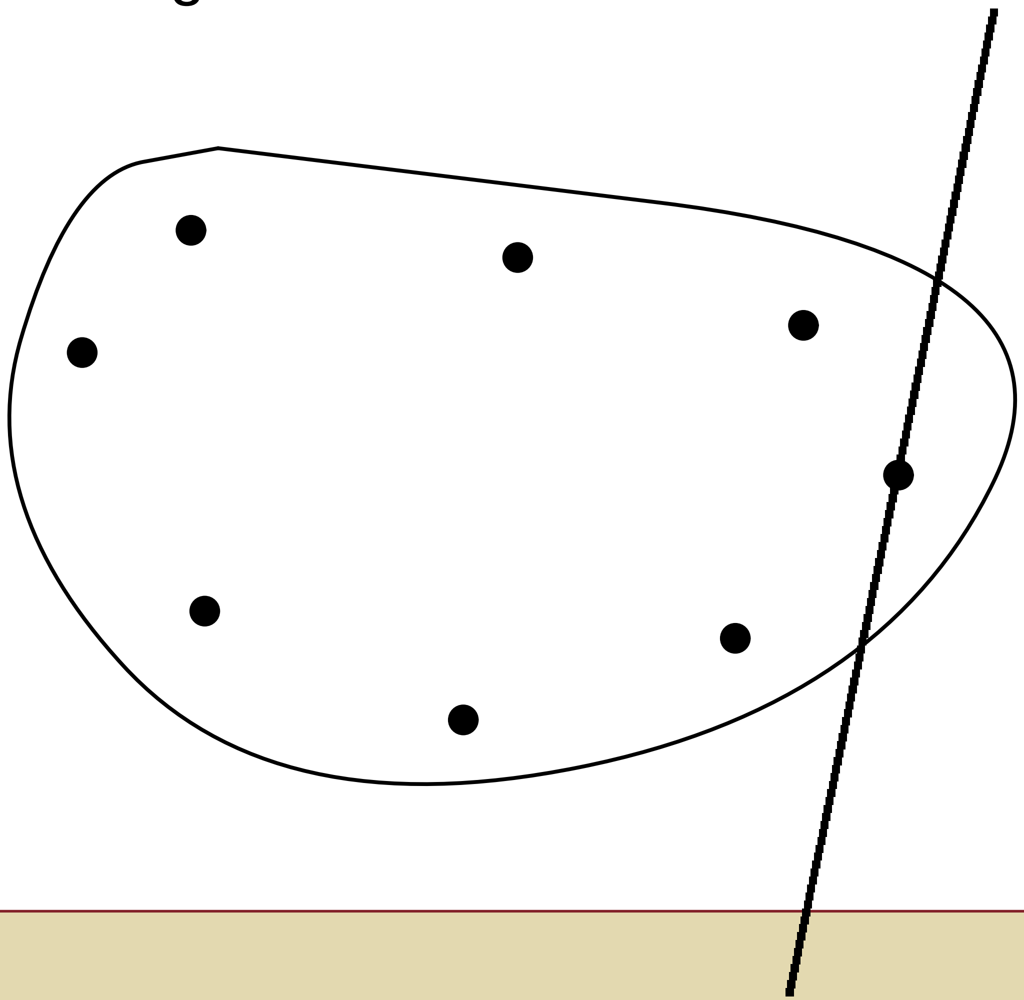

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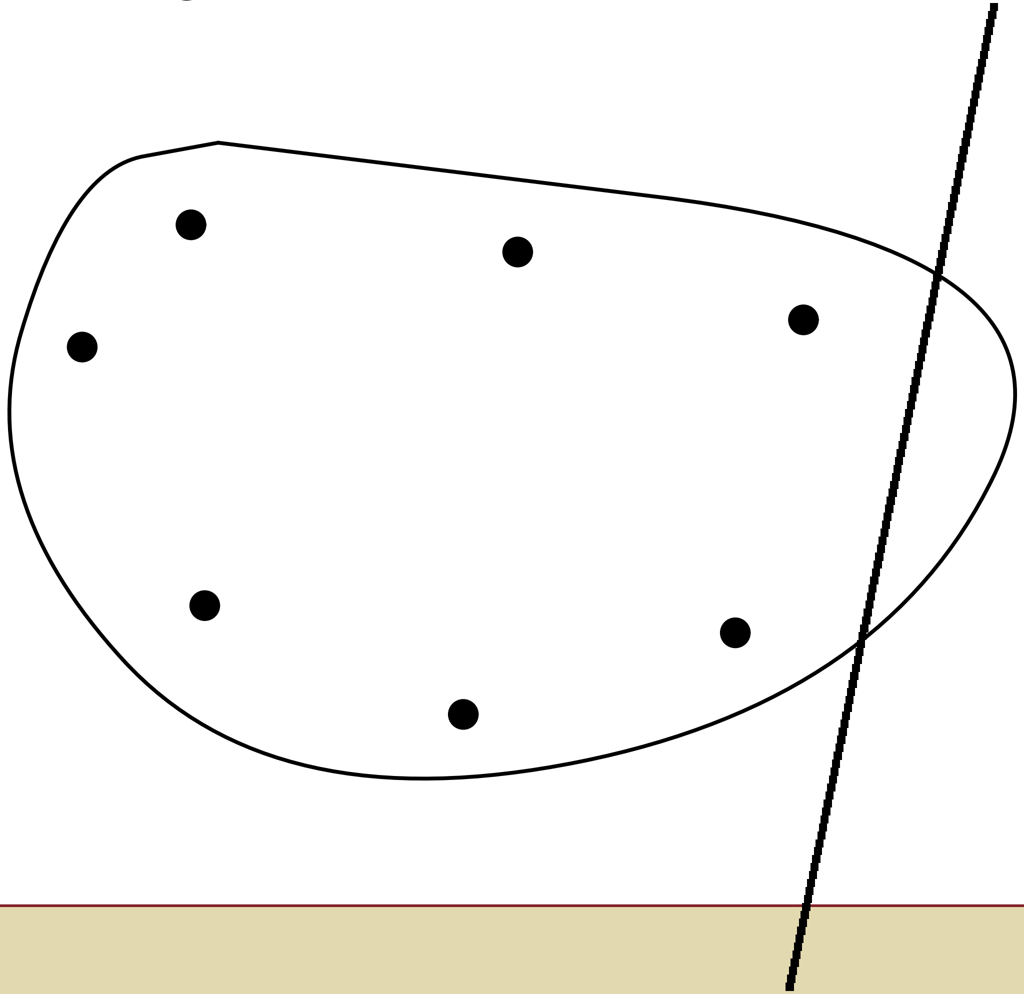

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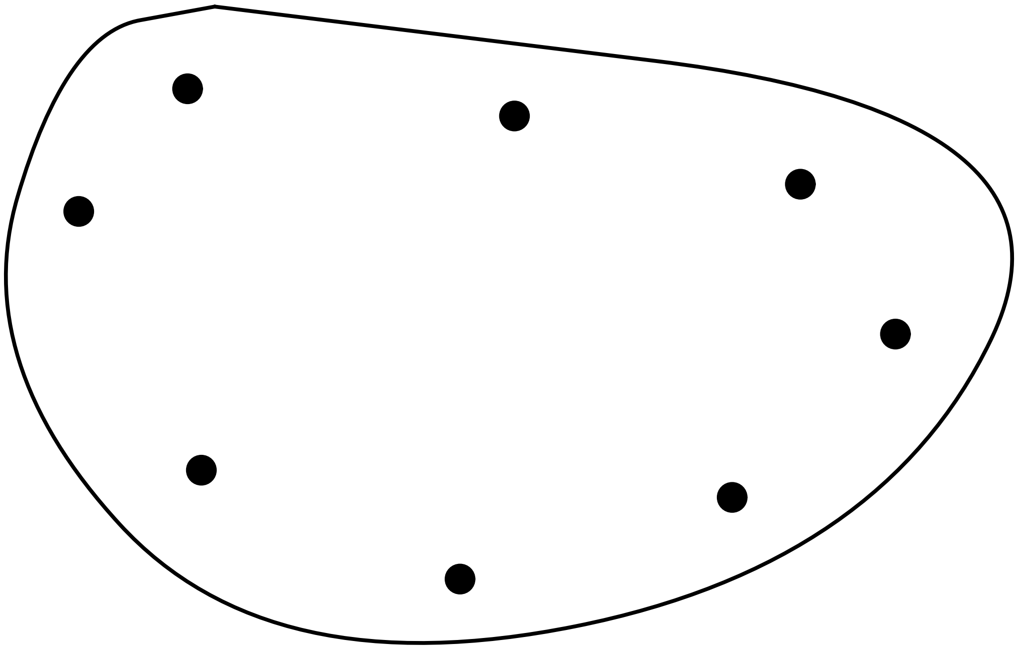

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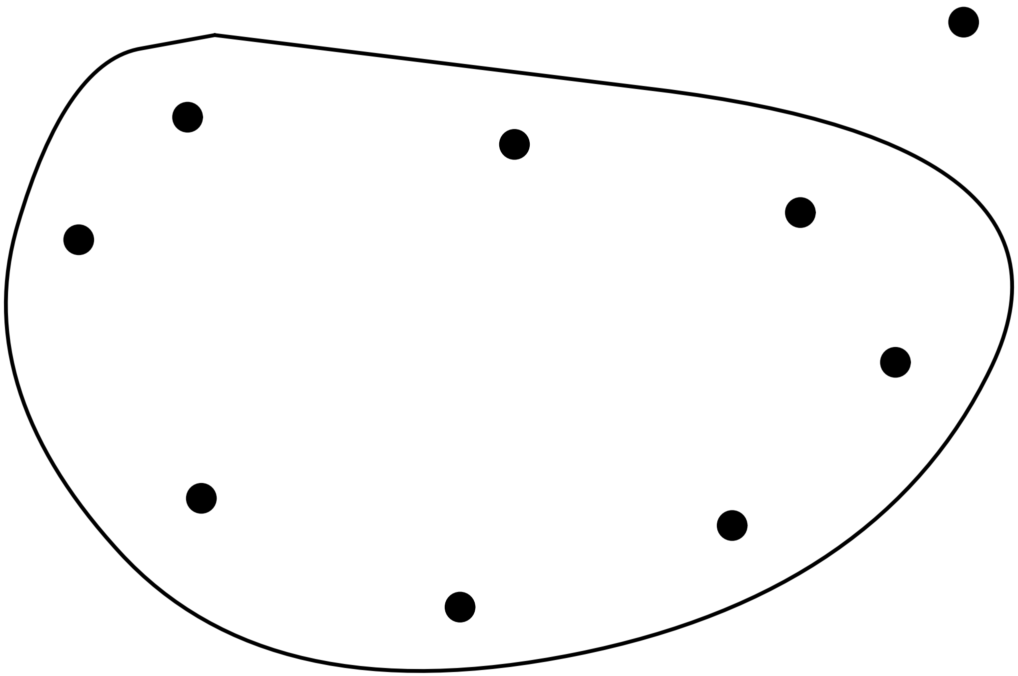

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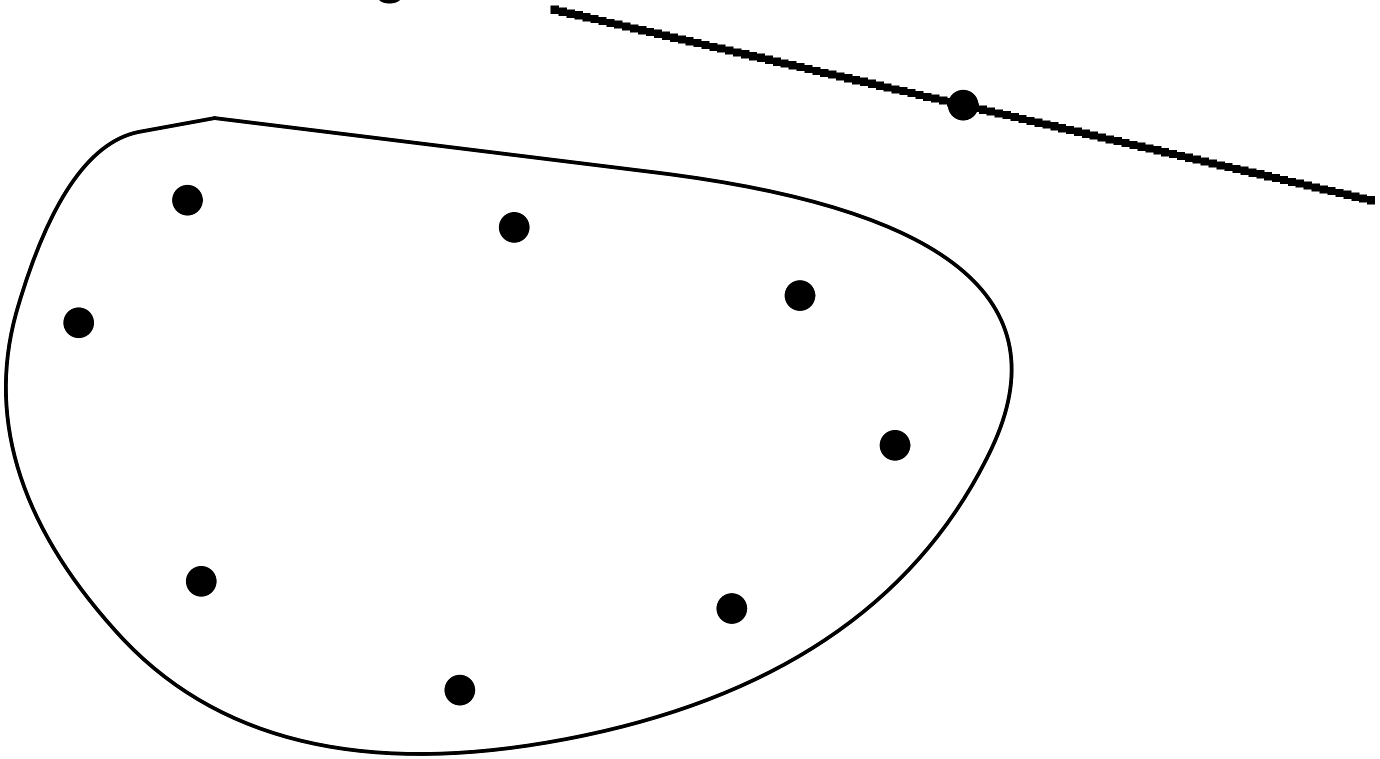

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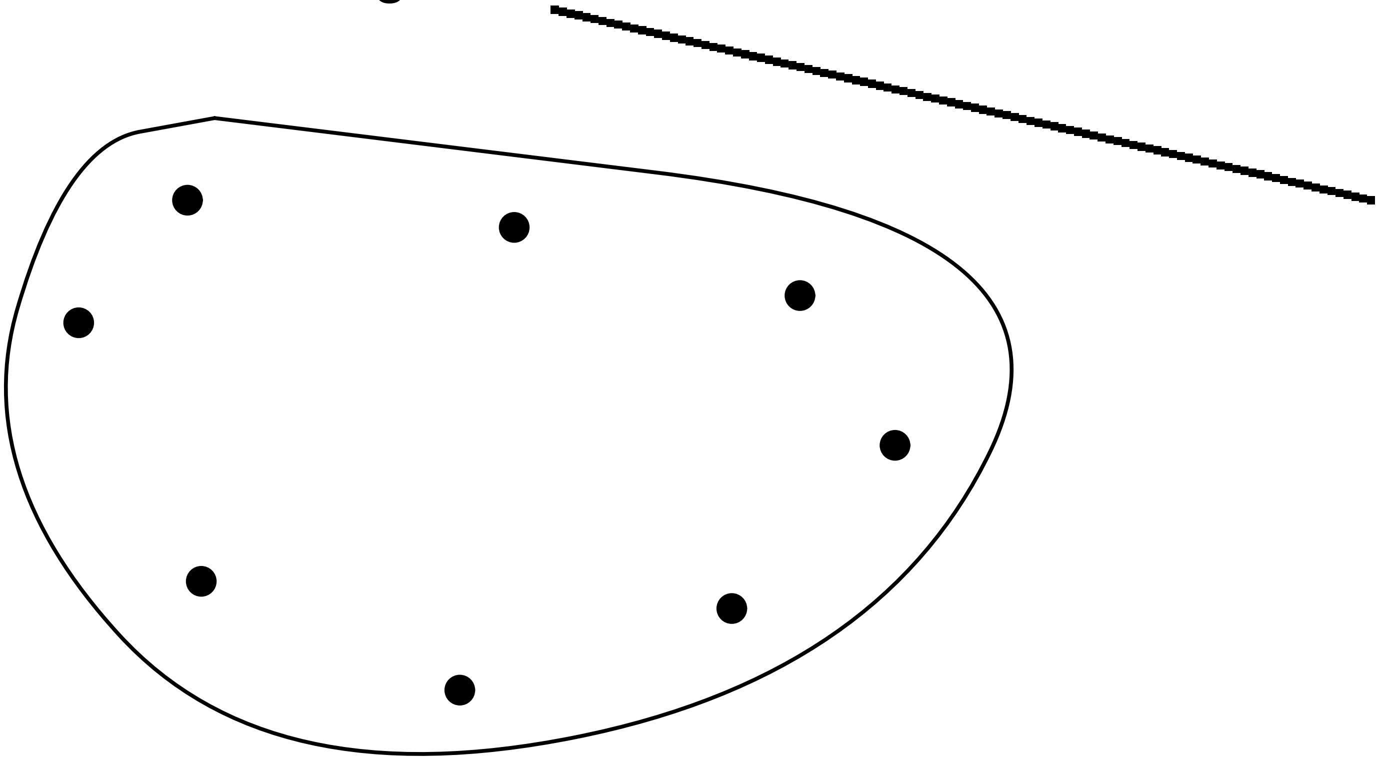

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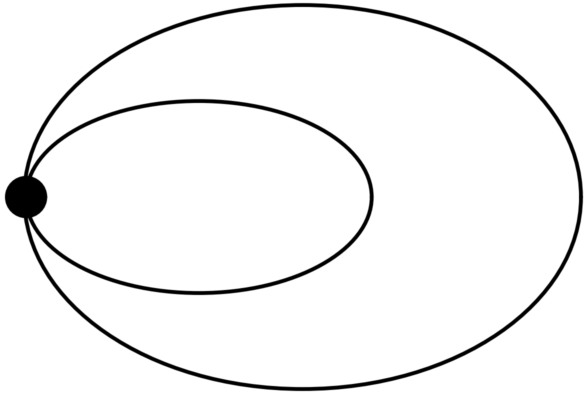
Some of our original motivation came from a 1992 paper by Tamás Szőnyi titled “Note on the existence of large minimal blocking sets in Galois planes.”

tangent pencil



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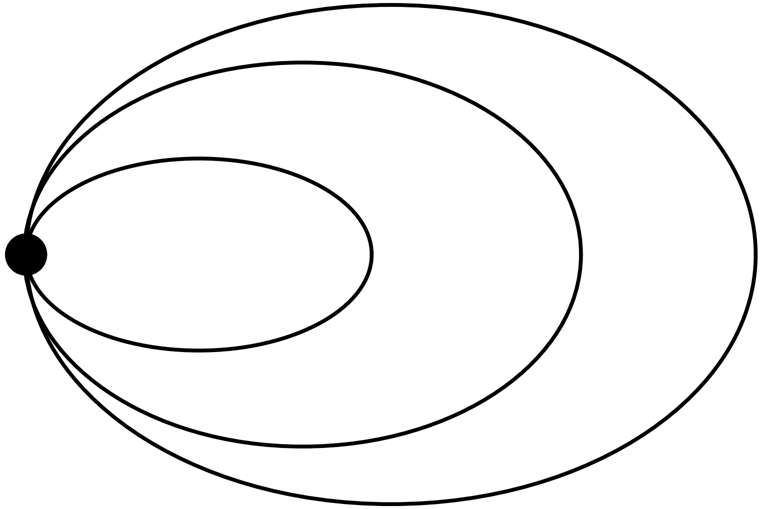

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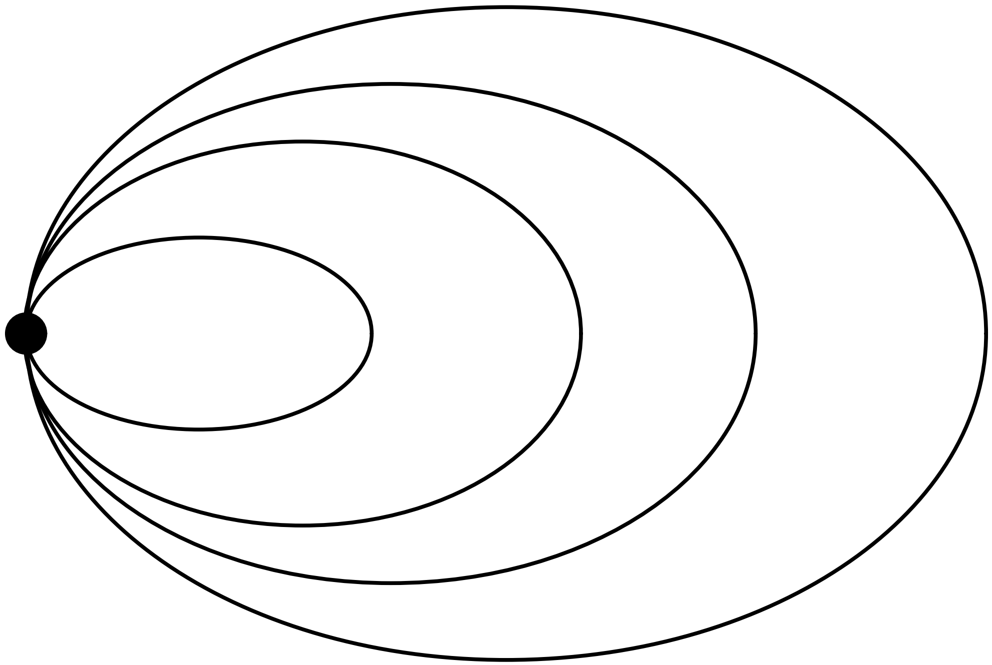

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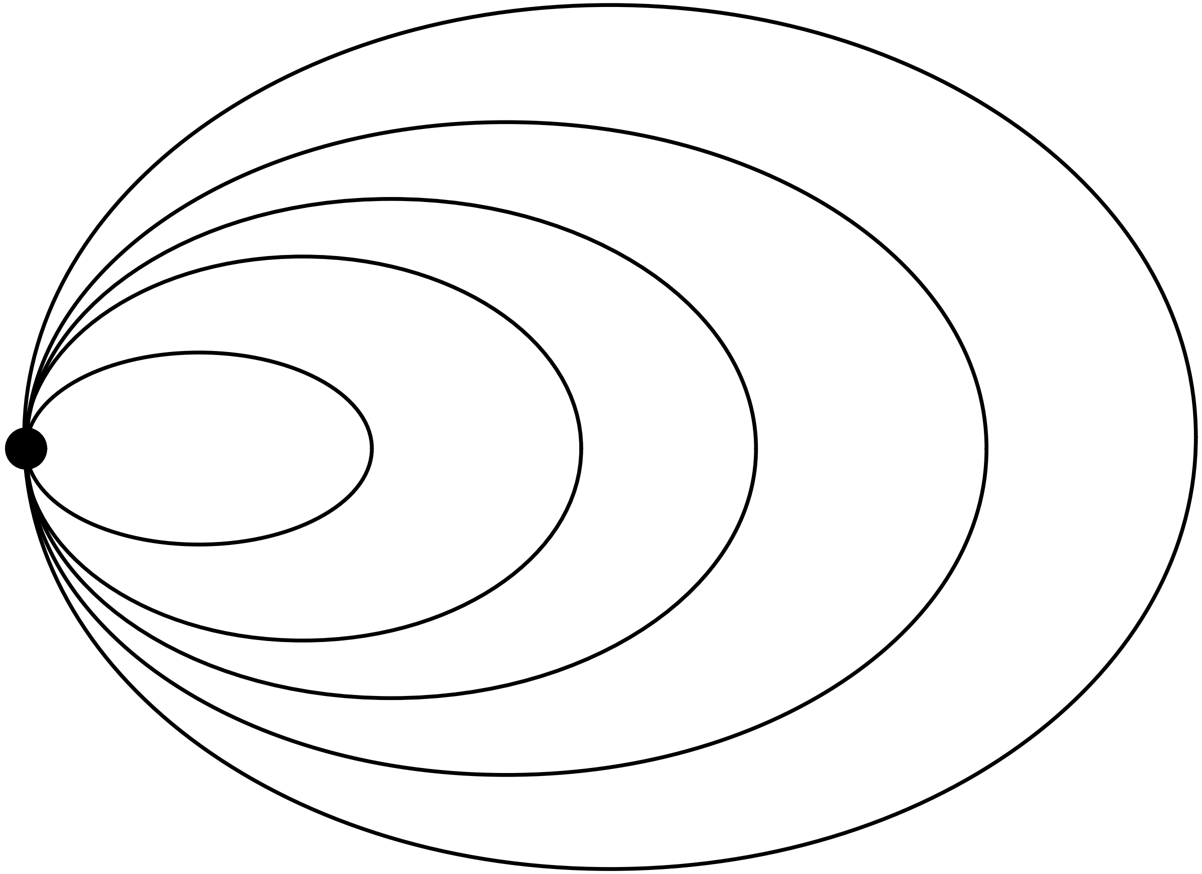

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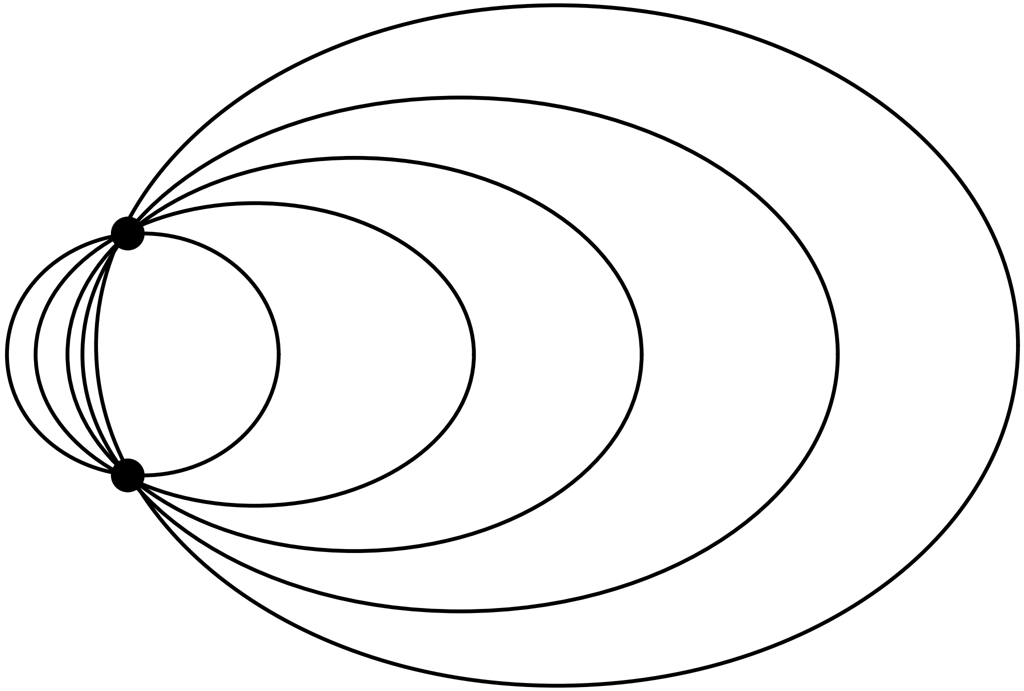
Was there any reason to start with a pair of tangent conics?

other pencils



Our Main Result

Theorem: Let $\mathcal{B} = \bigcup_{i=1}^{k} \mathcal{C}_i$ be a semioval in $PG(2, q)$ that is the union of nondegenerate conics $\mathcal{C}_i$. Then $\mathcal{B}$ is isomorphic to one of the following sets:

- a conic (note that this is the only possibility when q is even), or
- a union of at most $\sqrt{q}$ conics all lying in a common pencil, or
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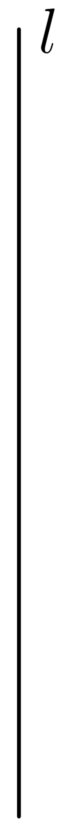
So the semiovals that can be written as a union of conics are completely classified.

What is a blocking set?

A blocking set $\mathcal{S}$ in a projective plane π is a set of points such that every line in π contains at least one point in $\mathcal{S}$ and one point not in $\mathcal{S}$. Therefore $\mathcal{S}$ intersects every line in π but $\mathcal{S}$ contains no lines.

Example #1



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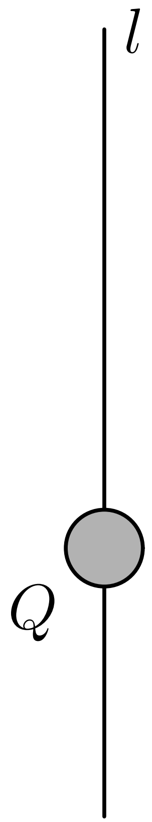

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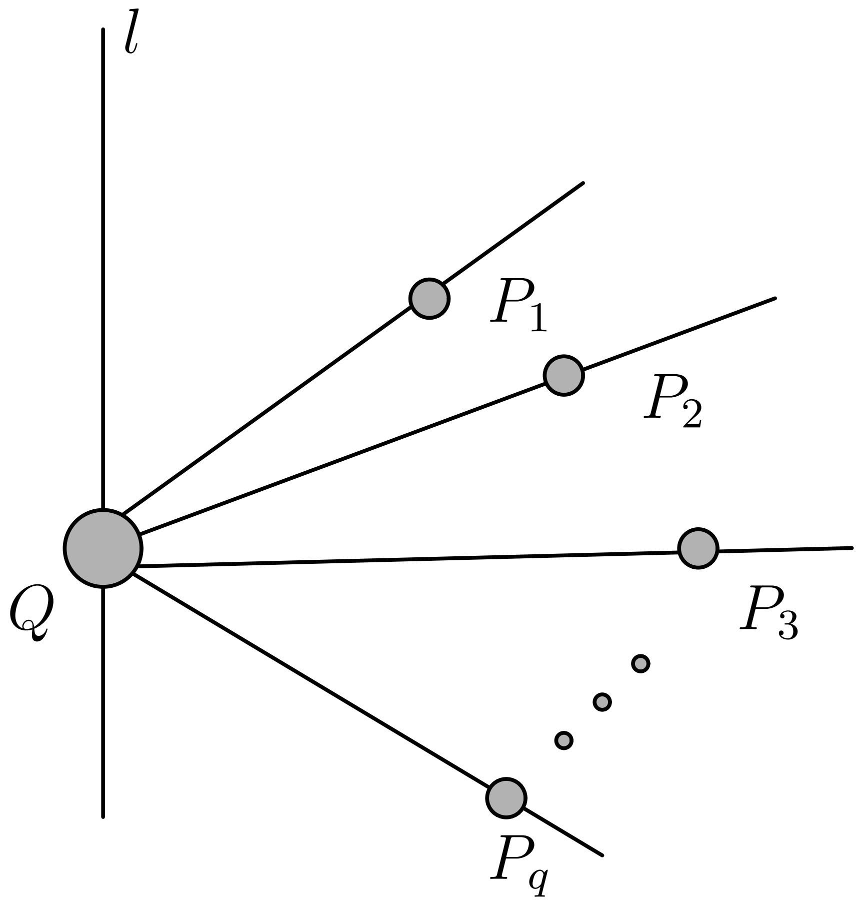
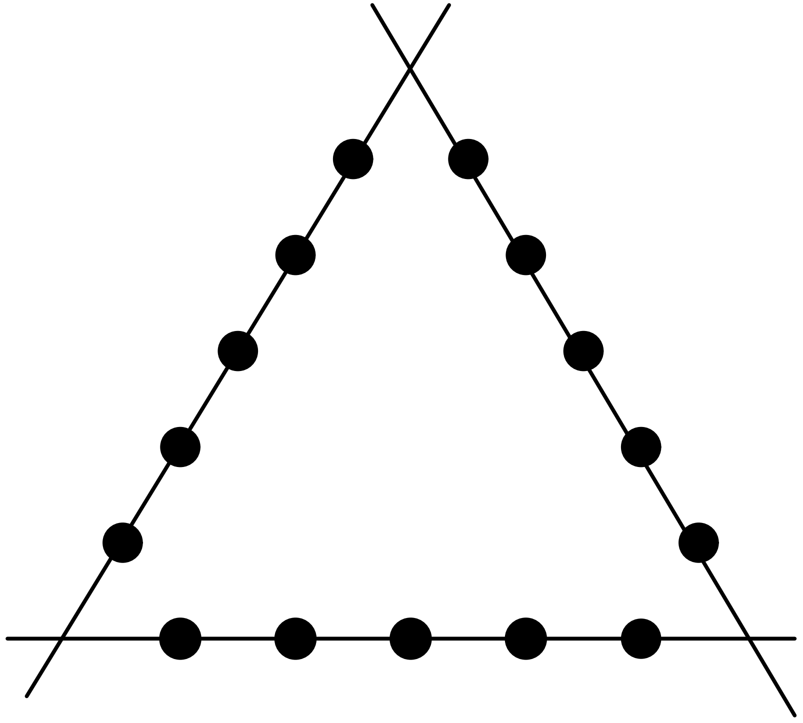


Example #2: Vertexless Triangle



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- is not a subset of any blocking set.

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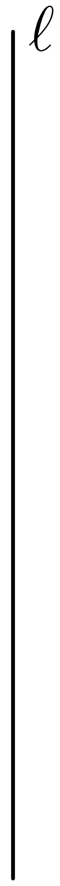

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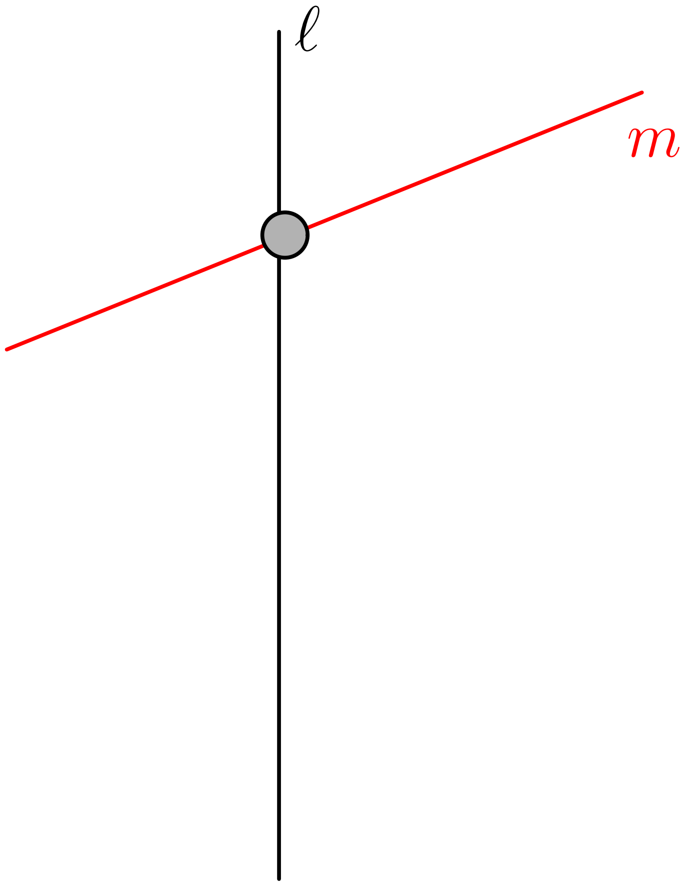

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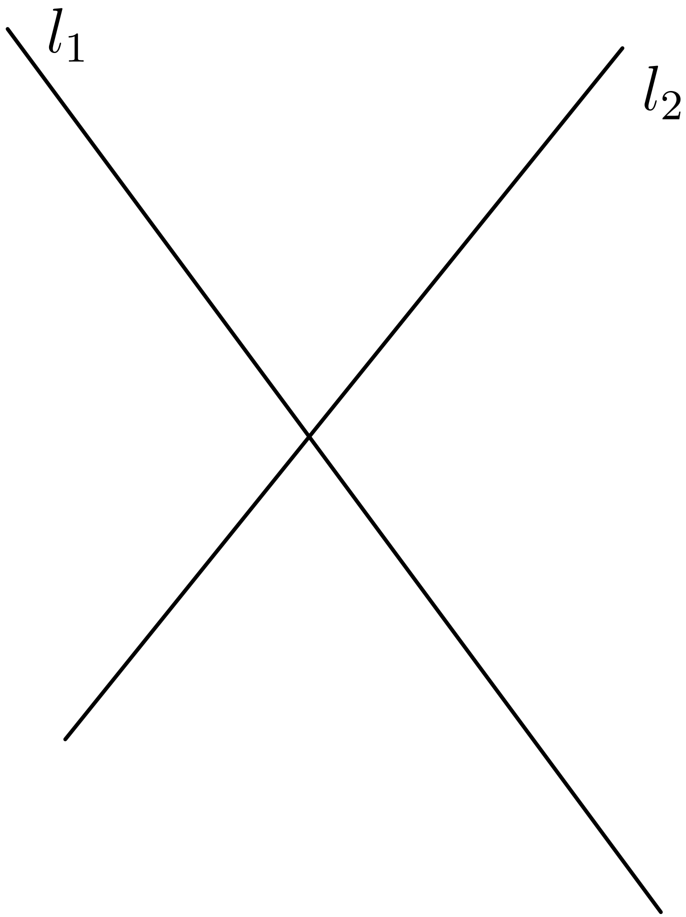

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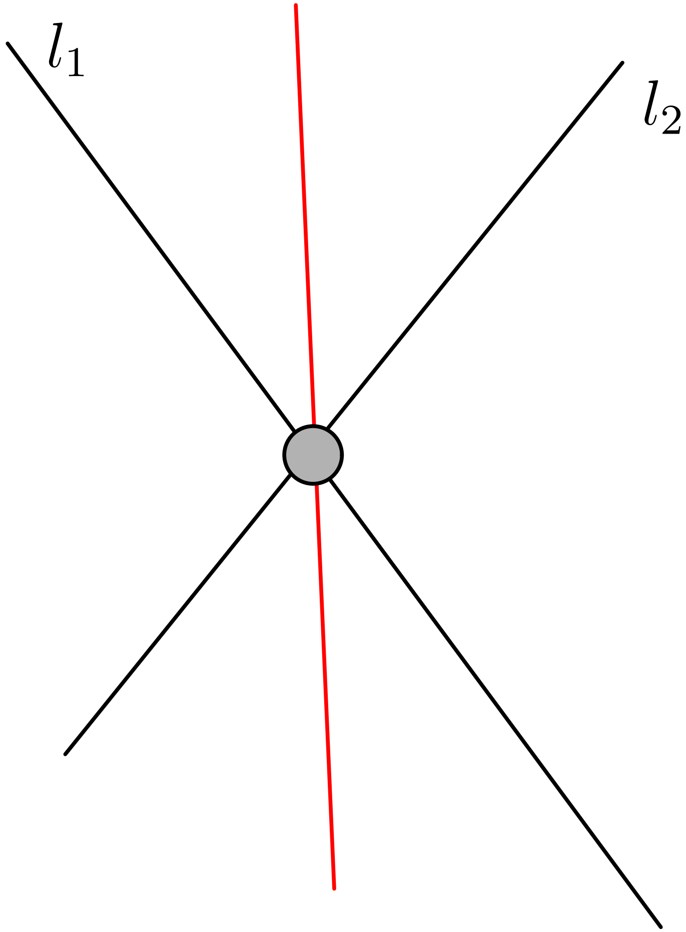

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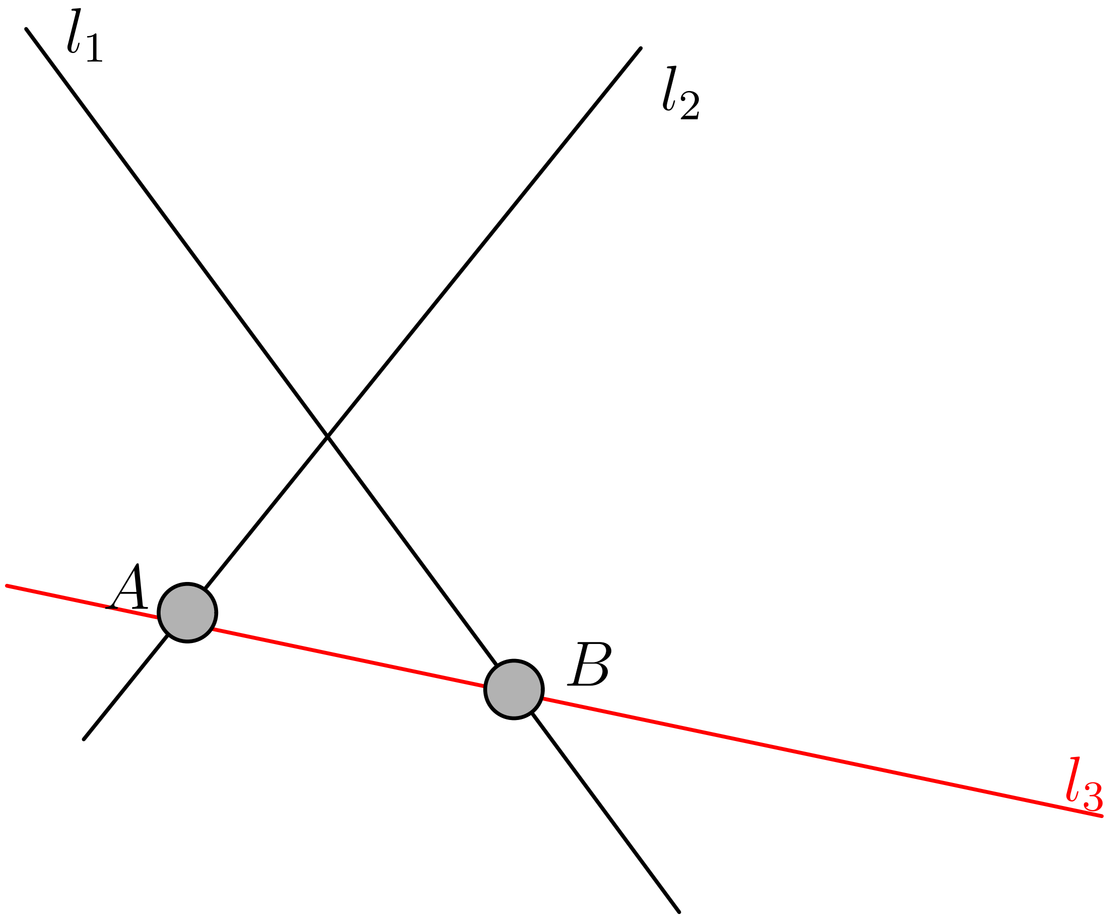
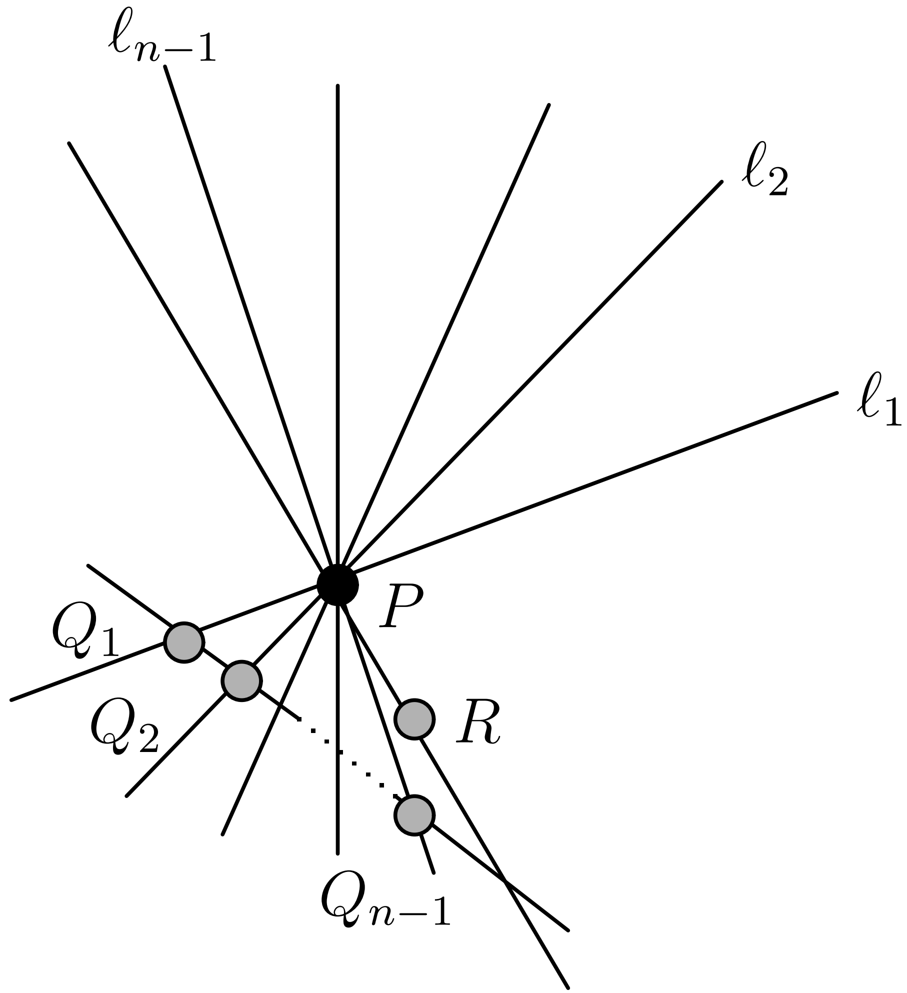


Results

Theorem Let $\mathcal{A}$ be an anti-blocking set and $\mathcal{L}$ the set of all lines that do not contain a point of $\mathcal{A}$. Then $|\mathcal{L}| \leq 2$.

By way of contradiction, suppose $|\mathcal{L}| \not\leq 2$ so that $|\mathcal{L}| \geq 3$. We show that points can be added to $\mathcal{A}$ in such a way as to build a blocking set. Therefore, $\mathcal{A}$ is in fact a subset of a blocking set and so is not an anti-blocking set. This will prove our result.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ℓ_2
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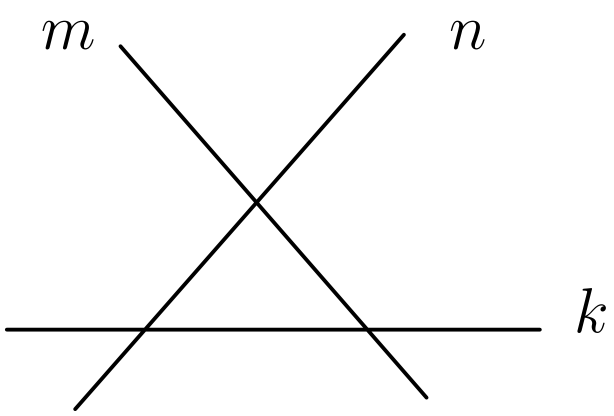
P
Q_1
Q_2
R
Q_{n-1}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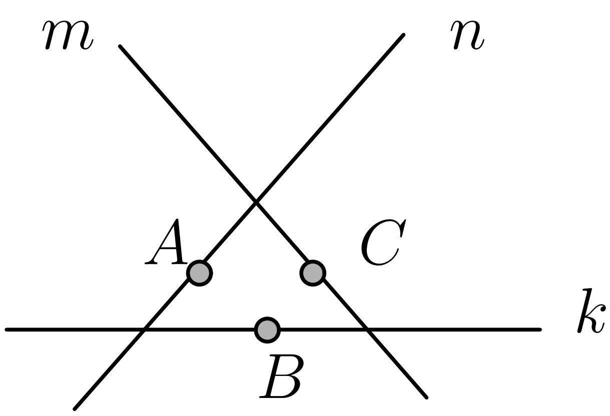
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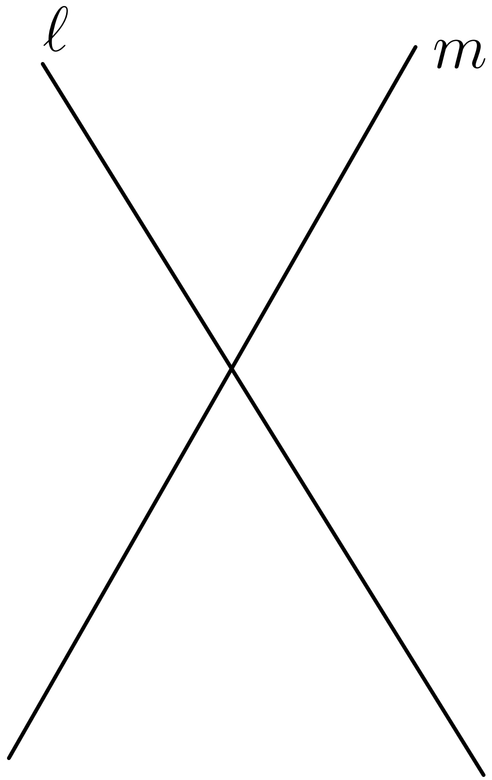

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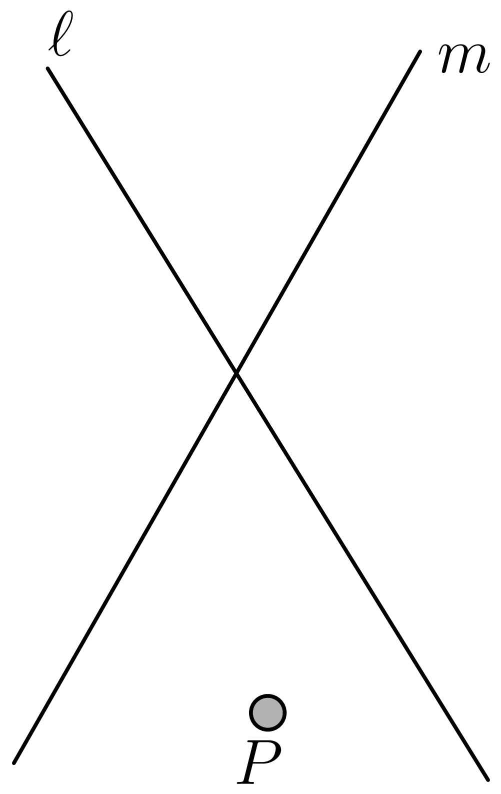

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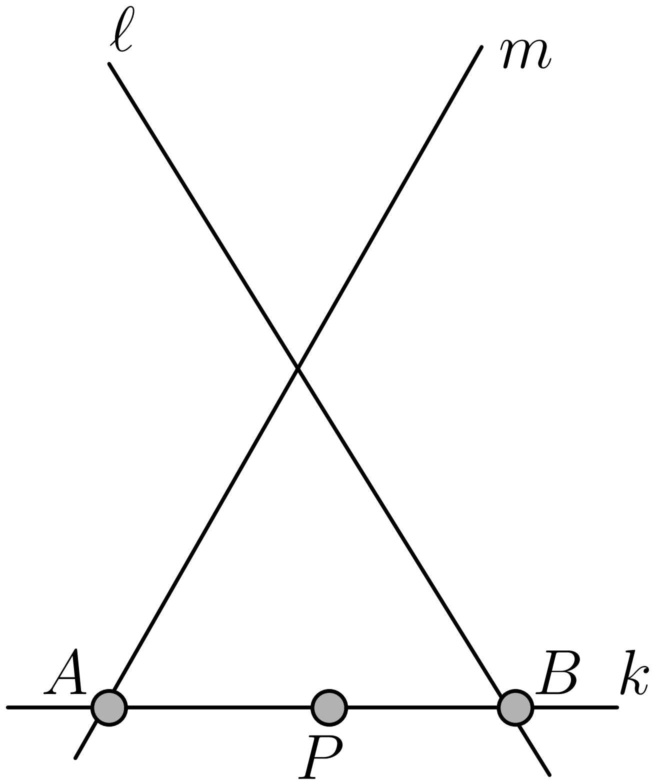

A characterization

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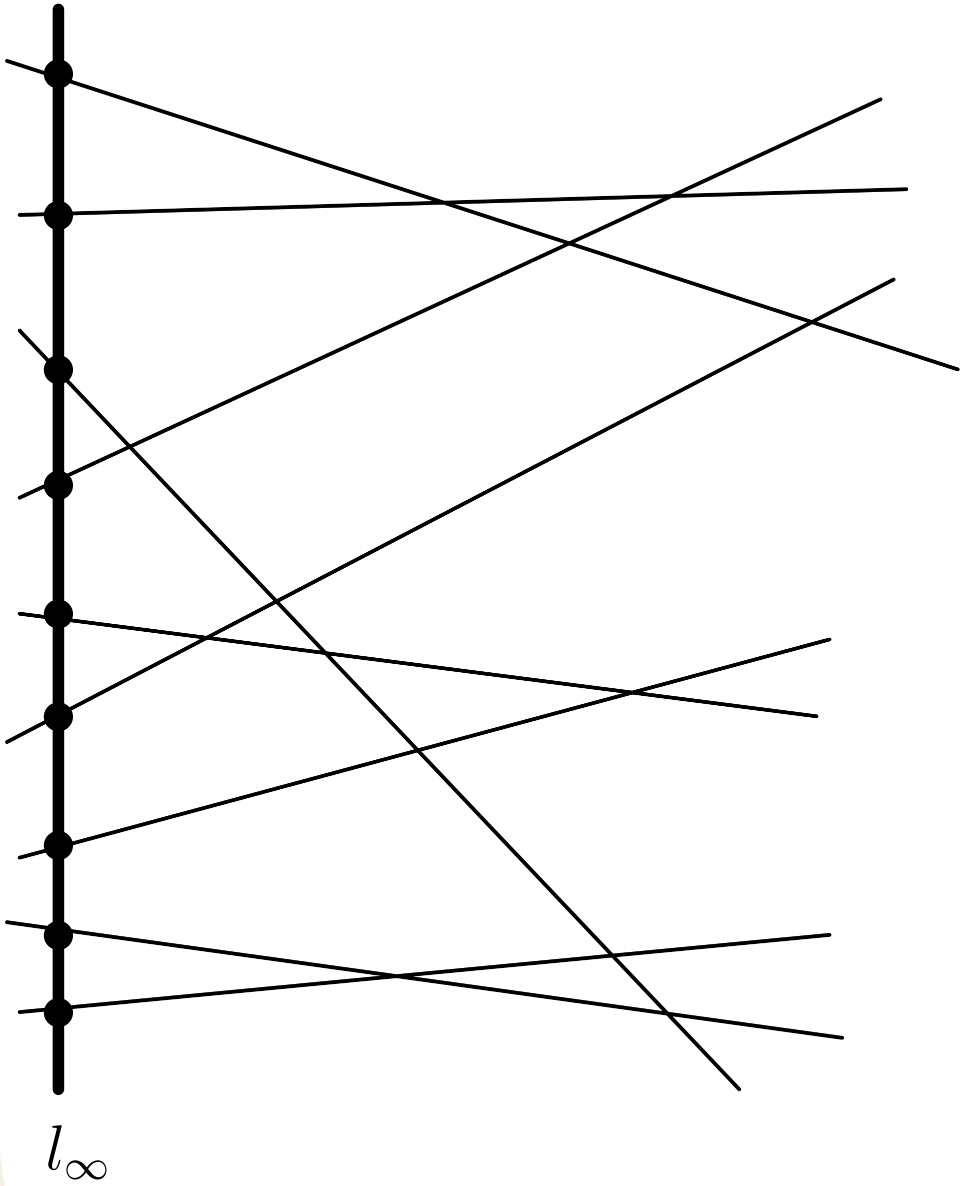
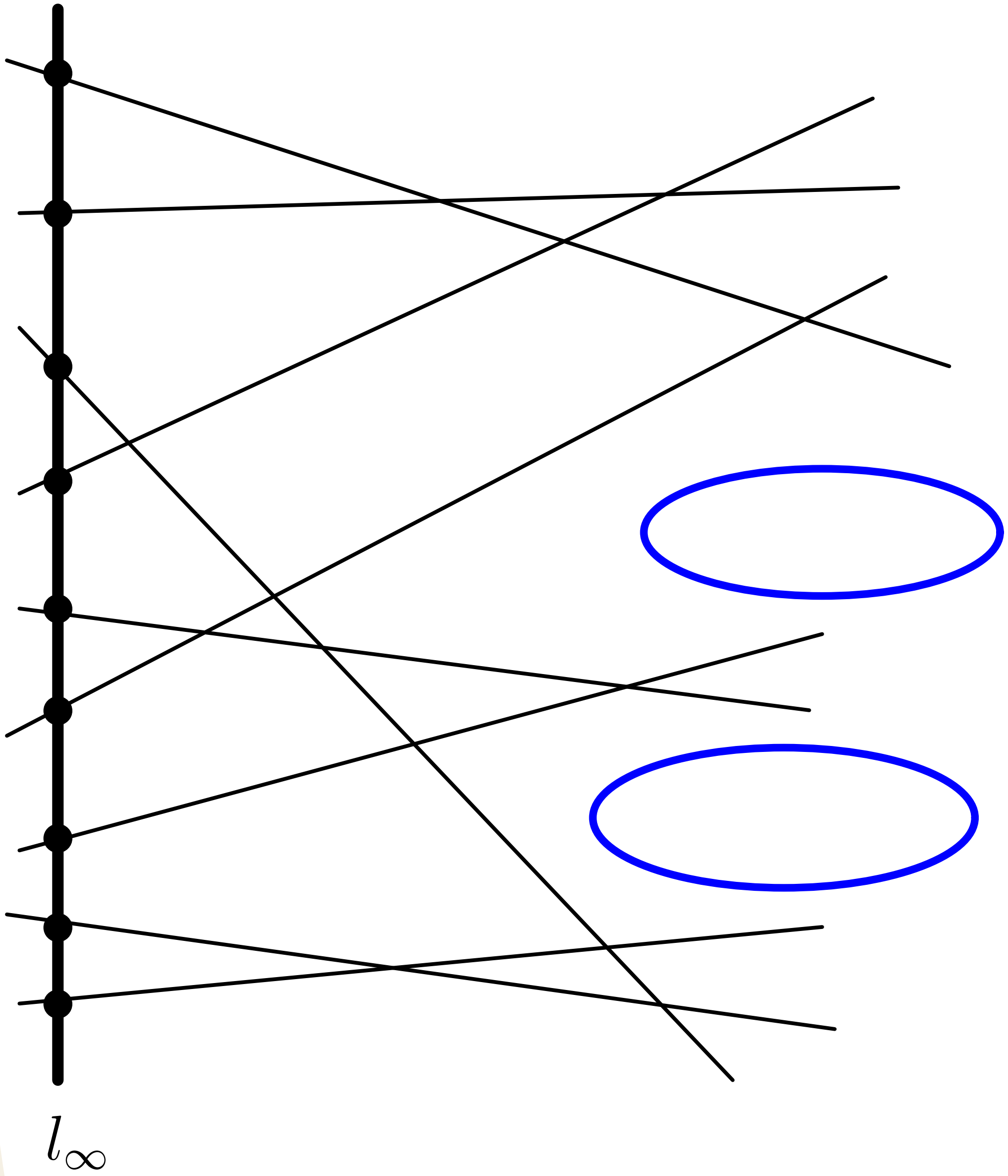
Theorem: Let ℓ_∞ be a line of the projective plane π. Suppose $\mathcal{A} \subseteq (\pi \setminus \ell_\infty)$, and $\mathcal{A}$ is not the complement of two lines. Then, $\mathcal{A}$ is an anti-blocking set if and only if for every point $P \in \ell_\infty$, there is a line ℓ ($\neq \ell_\infty$) through P such that $(\ell \setminus \{P\}) \subset \mathcal{A}$.

A characterization



Remove conics?



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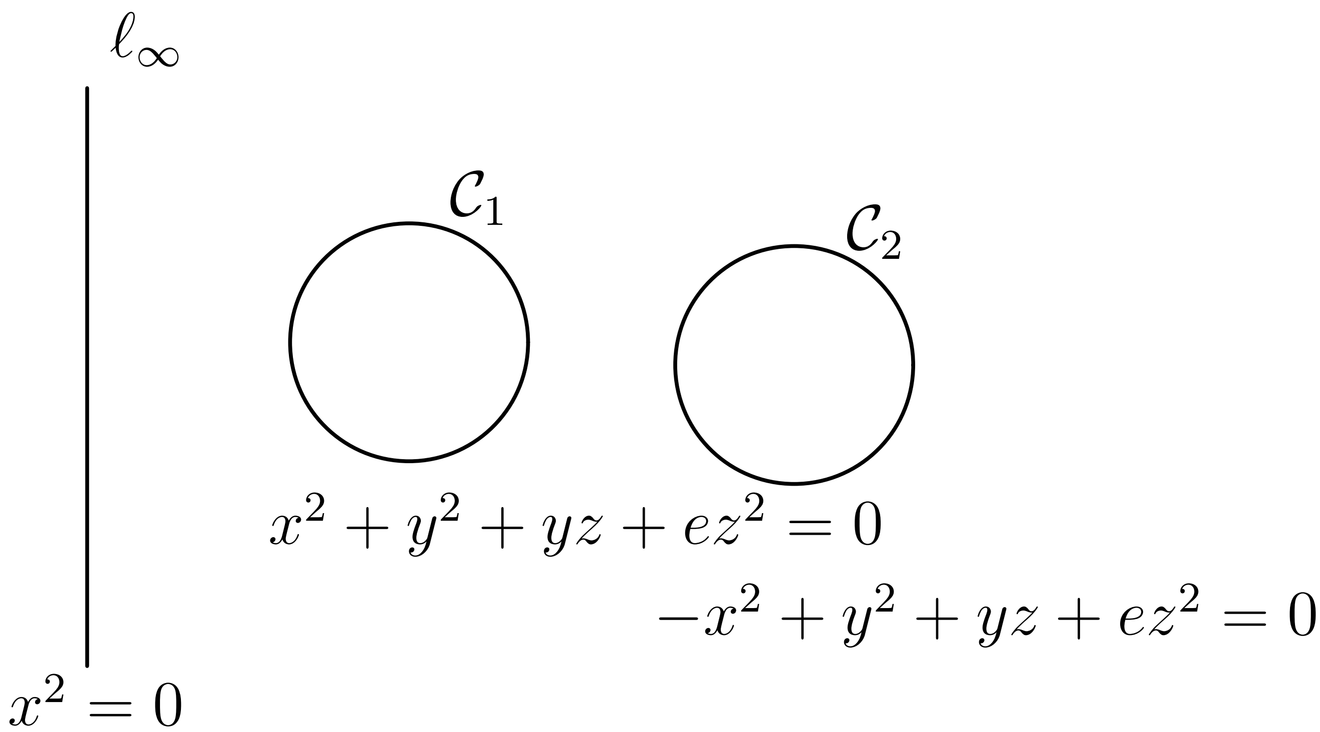

An anti-blocking sets from a pencil

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We argue that one can select $\frac{q-1}{4}$ conics from an algebraic pencil, remove them from the point-set of $\pi \setminus \ell_\infty$, and be left with a collection of points forming an anti-blocking set.

An anti-blocking sets from a pencil

If $P = (0, a, b)$ and $a^2 + ab + eb^2$ is a non-square, the special line is

$$\ell = \langle (0, a, b), (1, 0, 0) \rangle$$

If $P = (0, a, b)$ and $a^2 + ab + eb^2$ is a square, we have two lines:

$$\ell_+ = \langle (0, a, b), (1, 0, c) \rangle, \ell_- = \langle (0, a, b), (1, 0, -c) \rangle$$

An anti-blocking sets from a pencil

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If $P = (0, a, b)$ and $a^2 + ab + eb^2$ is a square, we have two lines:

$$\ell_+ = \langle (0, a, b), (1, 0, c) \rangle, \ell_- = \langle (0, a, b), (1, 0, -c) \rangle$$

The technique is all algebraic. We use the face that -1 is a square, and we select λ to have certain properties (which is guaranteed by the cyclotomic numbers of order 2).

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In a projective or affine plane of order q, the maximal size of a cap is bounded by $q+1$ if q is odd, or $q+2$ if q is even. These bounds are achieved in the Desarguesian and many other planes, though there do exist planes where the maximal cap is smaller than these bounds, for example the dual derived semifield plane of order 16.

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These bounds are non-constructive, and the best-known caps are significantly smaller than these bounds. In $AG(4, q)$, q even, the best known construction is due to Edel and Bierbrauer and yields a $(3q^2 + 4)$-cap.

The Pellegrino cap

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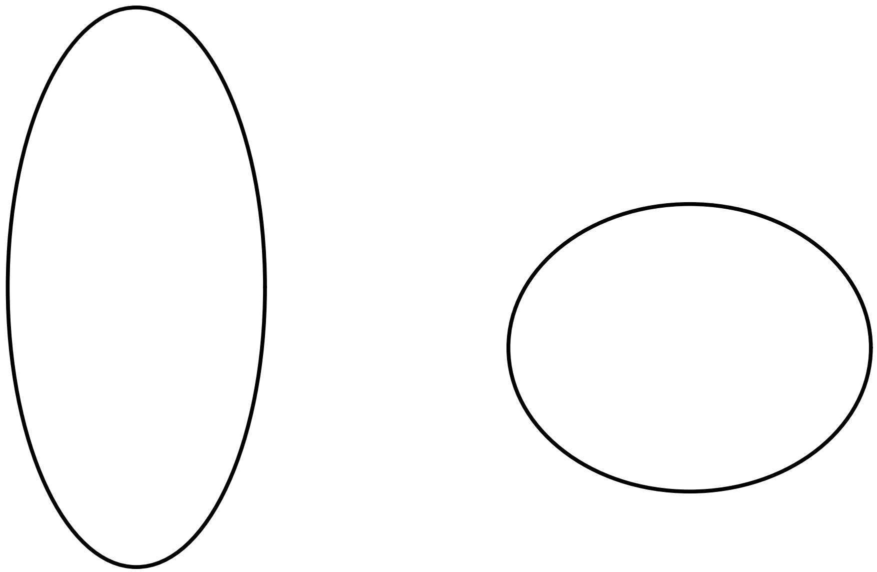
Looking at the image of a Pellegrino cap in $AG(4,3)$ under this bijection, Tucker noticed that the resulting set in $AG(2,9)$ is the union of two conics, each conic consisting exclusively of interior points of the other conic.

mutually interior conics



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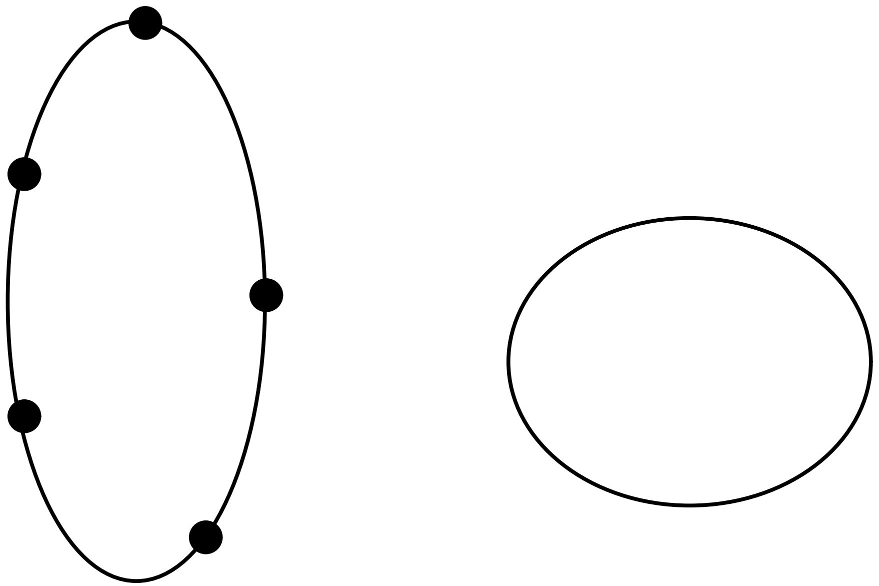

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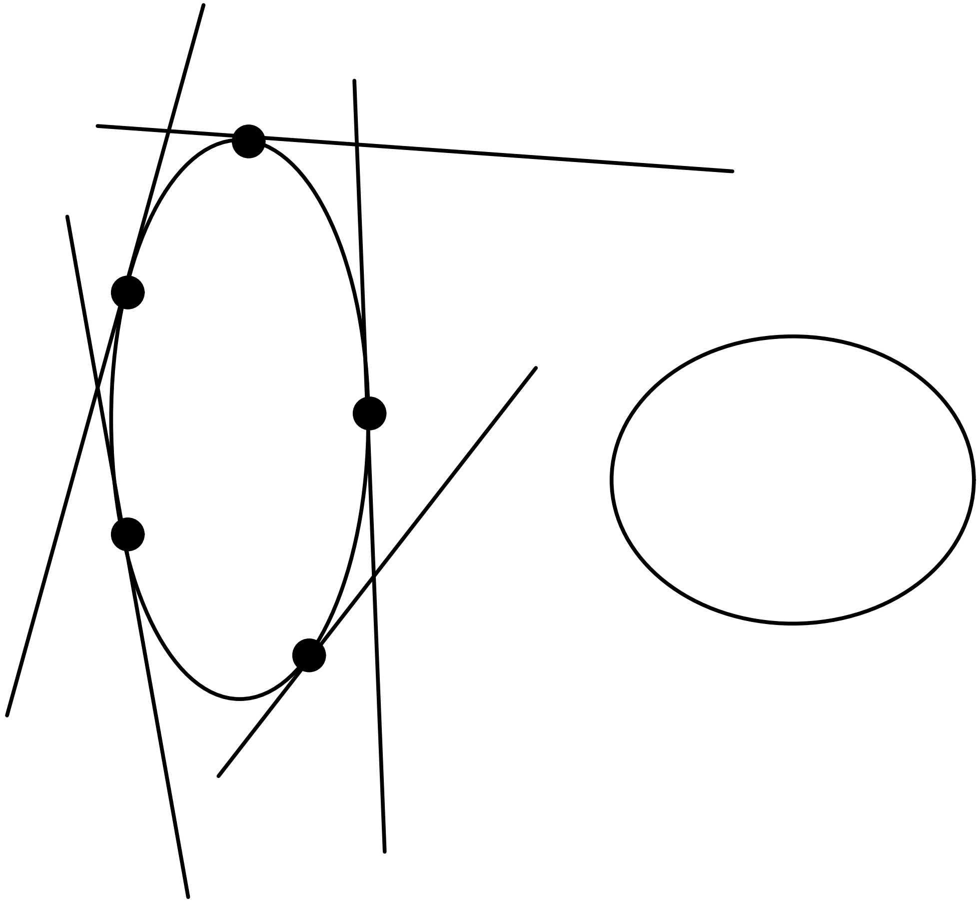

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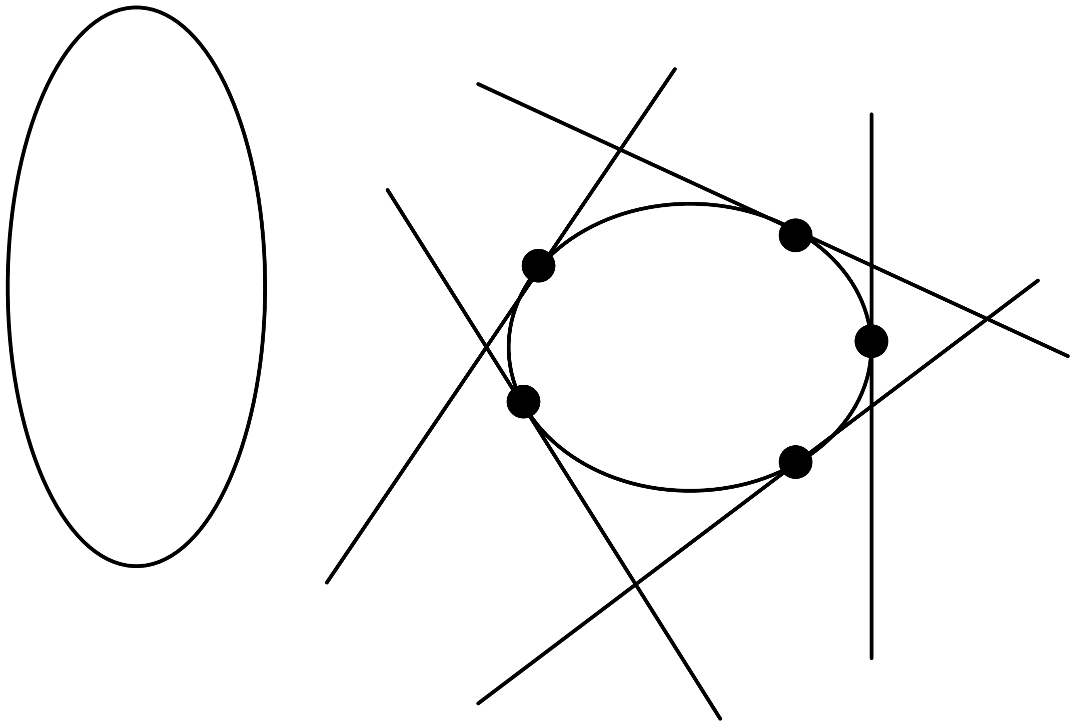

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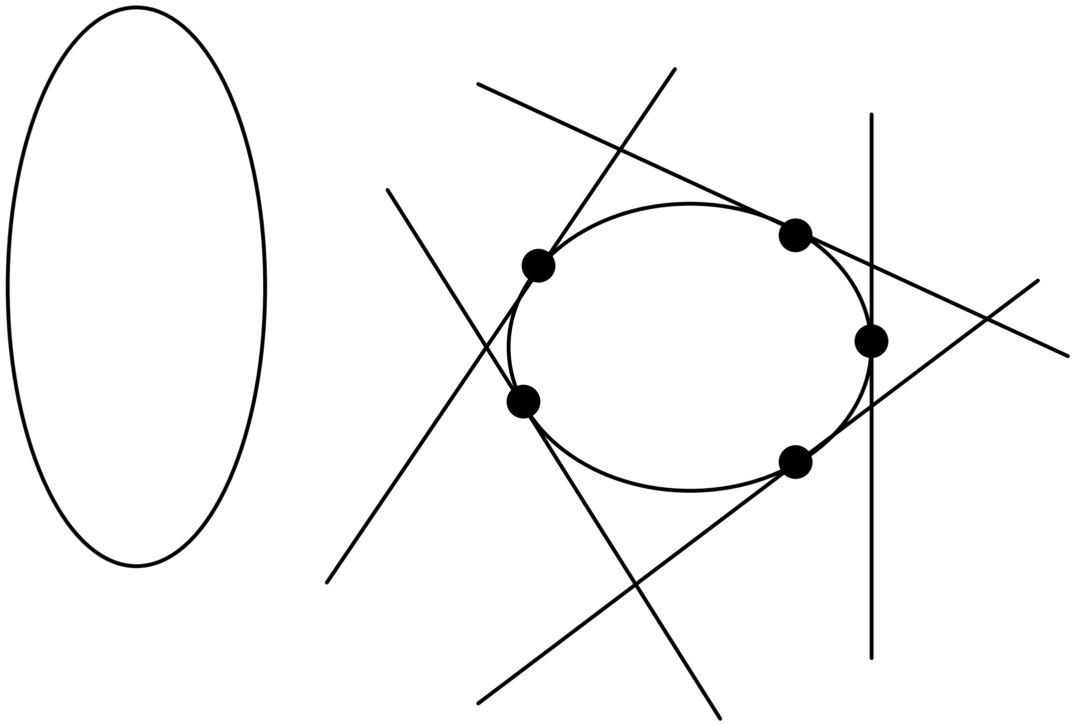
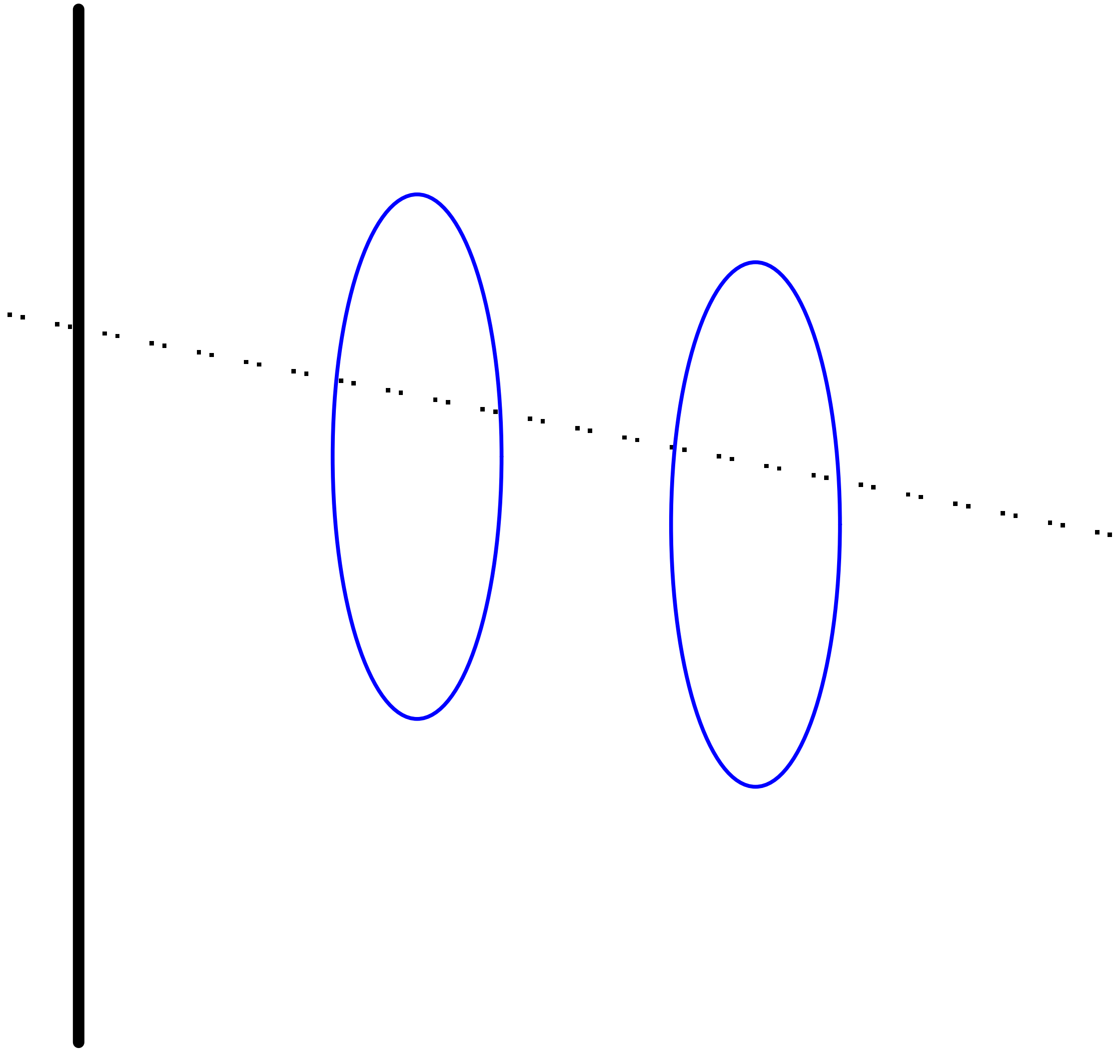


the conics are "mutually interior"

good pairs of conics



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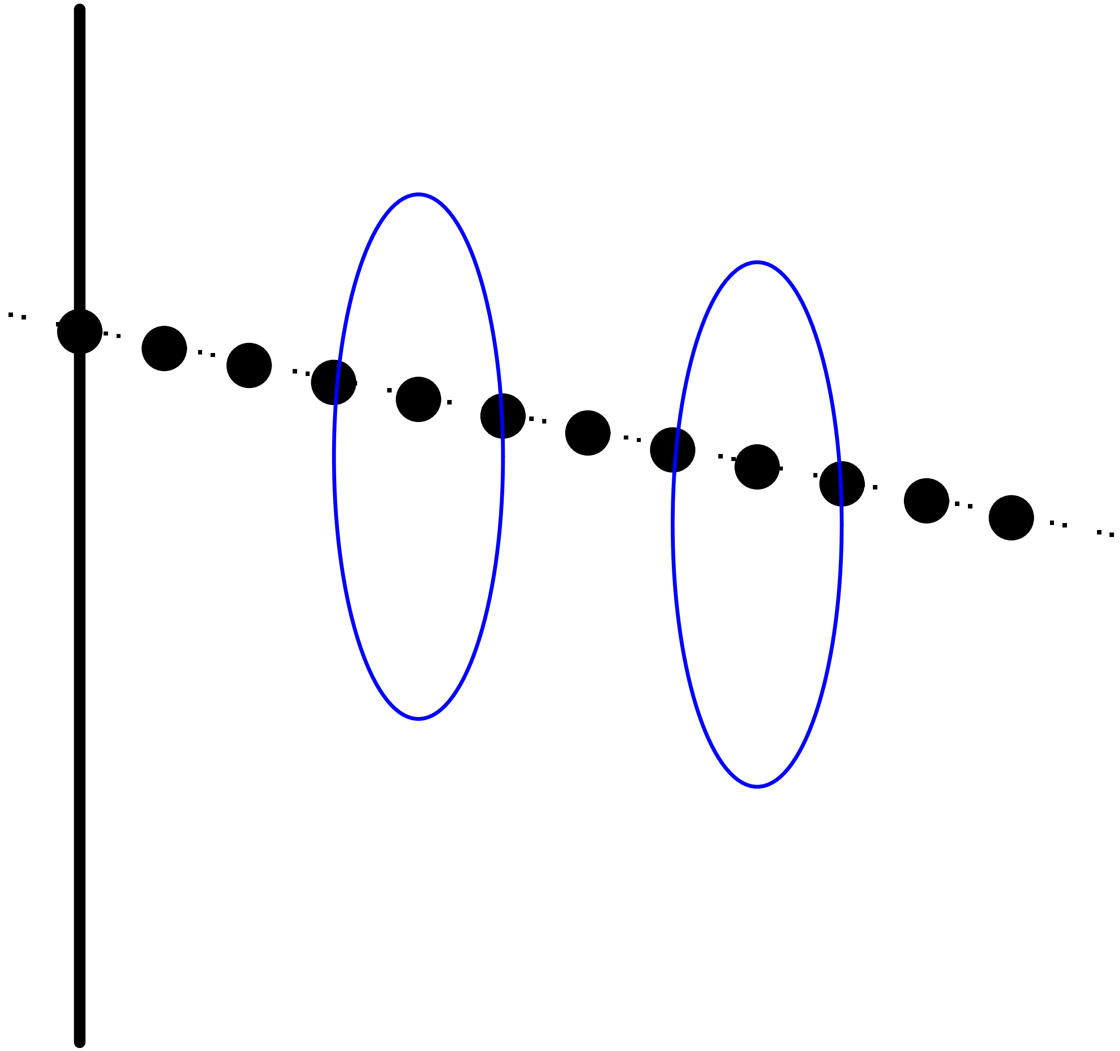

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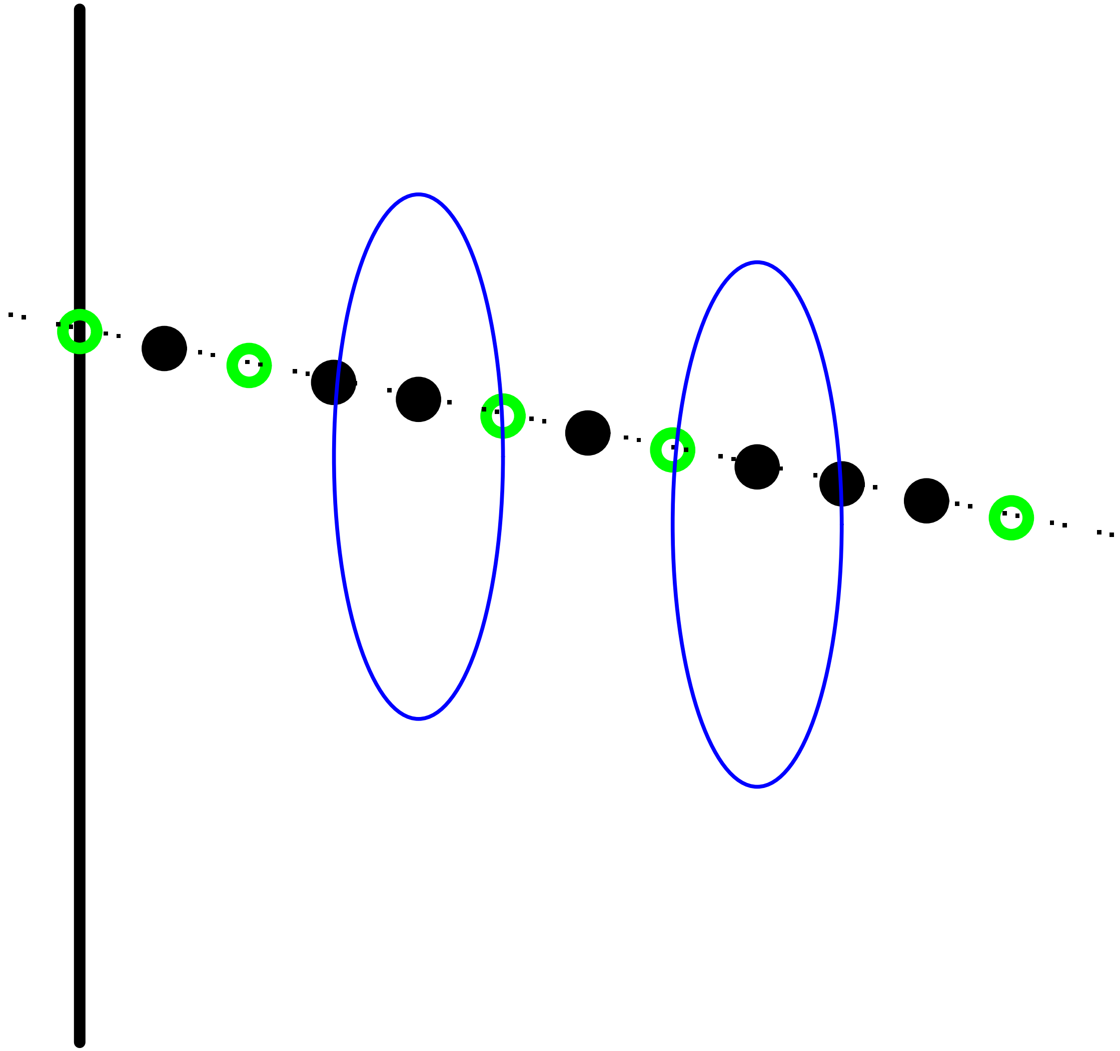

Mutually good ellipses

Let $\pi = AG(2, q^2)$, and let $\mathcal{P} = \{\mathcal{C}_\lambda : \lambda \in GF(q^2)\}$ be the linear pencil of conics with $\mathcal{C}_\lambda = \{(x, y) \in \pi : x^2 + xy + ey^2 = \lambda\}$, where $x^2 + x + e$ is irreducible.

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Theorem: The ellipse $\mathcal{C}_\lambda$, $\lambda \notin \{0, 1\}$, is $\mathcal{C}_1$-good if and only if for all $a, b \in GF(q^2)$ such that
$\frac{2a+b}{a^2+ab+eb^2} \in GF(q) \setminus \{-1, -2\}$,
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At this point, more can be said if we look at the even and odd cases separately.

q odd vs. q even

Theorem: Let q be an odd prime power and let λ be an element of the subfield $GF(q)$ of $GF(q^2)$ such that $\sqrt{\lambda}$ lies in $GF(q^2) \setminus GF(q)$. Then, $\mathcal{C}_\lambda$ is $\mathcal{C}_1$-good. Letting λ be an element of the subfield $GF(q)$ of $GF(q^2)$ such that $\sqrt{\lambda}$ lies in $GF(q^2) \setminus GF(q)$, $\mathcal{C}_1 \cup \mathcal{C}_\lambda$ is a Baer-cap.

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Theorem: Let q be an even prime power and suppose $\lambda^{q-1} = 1$ or $\lambda^{q+1} = 1$, with $\lambda \neq 1$. Then $\mathcal{C}_\lambda$ is $\mathcal{C}_1$-good. Letting $\lambda \neq 1$ satisfy either $\lambda^{q-1} = 1$ or $\lambda^{q+1} = 1$, $\mathcal{C}_1 \cup \mathcal{C}_\lambda$ is a Baer-cap. Moreover in the case where $\lambda^{q+1} = 1$, $\mathcal{C}_1 \cup \mathcal{C}_\lambda \cup \{(0,0)\}$ is a Baer-cap.

a special case

Let $q \equiv -1 \pmod 4$. Then, the set $\mathcal{C}_1 \cup \mathcal{C}_{-1}$ forms a pair of mutually good conics in $PG(2, q^2)$. Moreover, the group induced by matrices of the form

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A transitive group acting on the conics will induce a transitive action on the lifted set of points in $AG(4, q)$. Therefore, the cap admits a transitive automorphism group.

mutually good parabolas

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It follows that $AG(4,q)$, q odd, has a cap of size $2q^2$ lifted from two mutually good parabolas of $AG(2,q^2)$.

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Theorem: Let q be an odd power of 2, so that $q \equiv 2 \pmod 3$. Let $\lambda \neq 1$ be a cube root of unity in $GF(q^2)$. Then $\mathcal{C}_1 \cup \mathcal{C}_\lambda \cup \mathcal{C}_{\lambda^2}$ is a Baer-cap.

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The caps obtained here are almost certainly identical to the caps discovered by Edel and Bierbrauer, though their construction originated from a coding-theoretic perspective.

References

J. M. Dover and K. E. Mellinger. Semiovals from unions of conics. *Innov. Incidence Geom.*, to appear.

K. Scott. *Anti-blocking Sets.* UMW Honors Thesis, May 2012.

J. M. Dover, K. E. Mellinger, and K. L. Wantz, Generalized Pellegrino caps. *submitted.*

Conclusions

Questions?