Kirkman's Schoolgirls
wearing hats and walking through fields of numbers

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The Kirkman Schoolgirls Problem

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Every day, they walk to school in the Official ENBS Formation, namely, in five rows of three each. One of the ENBS rules is that during the walk, a student may only talk with the other students in her row of three.

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These fifteen are all good friends and like to talk with each other – and they are all mathematically inclined.

The Kirkman Schoolgirls Problem

One day Julia says, "I wonder if it's possible for us to walk to school in the Official Formation in such a way that we all have a chance to talk with each other at least once a week?"

The Kirkman Schoolgirls Problem

"But that means nobody walks with anybody else in a line more than once a week," observes Anita.

The Kirkman Schoolgirls Problem

"I'll bet we can do that," concludes Lori.
"Let's get to work."

Solution

MON	TUE	WED	THU	FRI	SAT	SUN
a, b, e	a, c, f	a, d, h	a, g, k	a, j, m	a, n, o	a, i, l
c, l, o	b, m, o	b, c, g	b, h, l	b, f, k	b, d, i	b, j, n
d, f, m	d, g, n	e, j, o	c, d, j	c, i, n	c, e, k	c, h, m
g, i, j	e, h, i	f, l, n	e, m, n	d, e, l	f, h, j	d, k, o
h, k, n	j, k, l	i, k, m	f, i, o	g, h, o	g, l, m	e, f, g

T. P. Kirkman

Rev. Thomas Penyngton Kirkman (31 March 1806 - 3 February 1895) was a British mathematician.

T. P. Kirkman

The problem first appeared in 1850 in the unlikely-sounding *Lady's and Gentlemen's Diary*, it reads as follows:

"Fifteen young ladies of a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk abreast more than once."

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his fame outside the field rests entirely on the Schoolgirls Problem and his solution

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4. Recreational mathematics

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For the Kirkman problem, we want $v = 15$, $k = 3$, and every set of 2 girls appears together in a block (or a row of the formation) exactly 1 time.

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For the Kirkman problem, we want $v = 15$, $k = 3$, and every set of 2 girls appears together in a block (or a row of the formation) exactly 1 time.

We say that a solution to the Kirkman problem is a $2 - (15, 3, 1)$ design

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When the block can be broken up like this, we say the design is *resolvable.*

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Mathematical formulation of the Kirkman Schoolgirls Problem:

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Find a resolvable $2 - (15, 3, 1)$ design containing 35 blocks.

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In fact, every element of this field can be written as $a + b\sqrt{2}$ for some rational numbers a and b (i.e., $a, b \in \mathbb{Q}$).

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$$(a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2}) = (a_1a_2 + 2b_1b_2) + (a_1b_2 + a_2b_1)\sqrt{2}$$

Since $a_1a_2 + 2b_1b_2 \in \mathbb{Q}$ and $a_1b_2 + a_2b_1 \in \mathbb{Q}$, we have shown closure.

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Notice that by letting $c = d = 0$, we have all of the elements of our smaller field.

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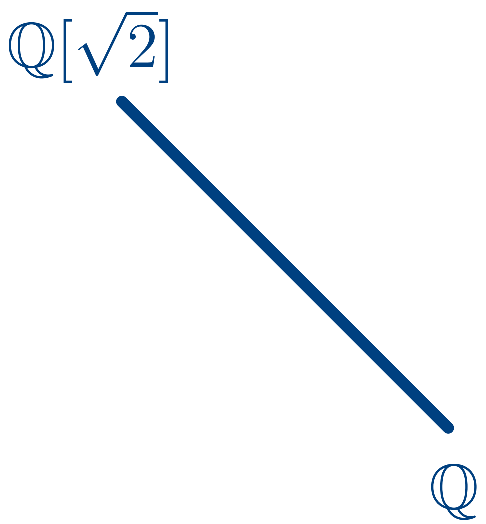

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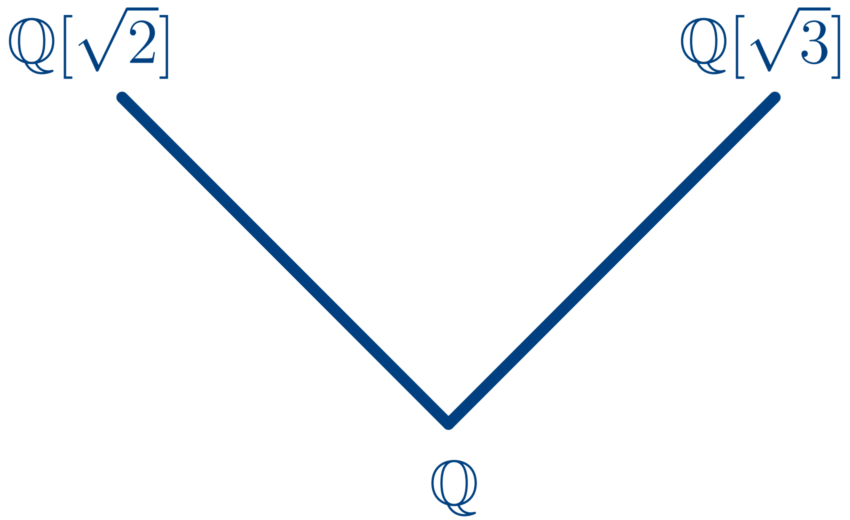

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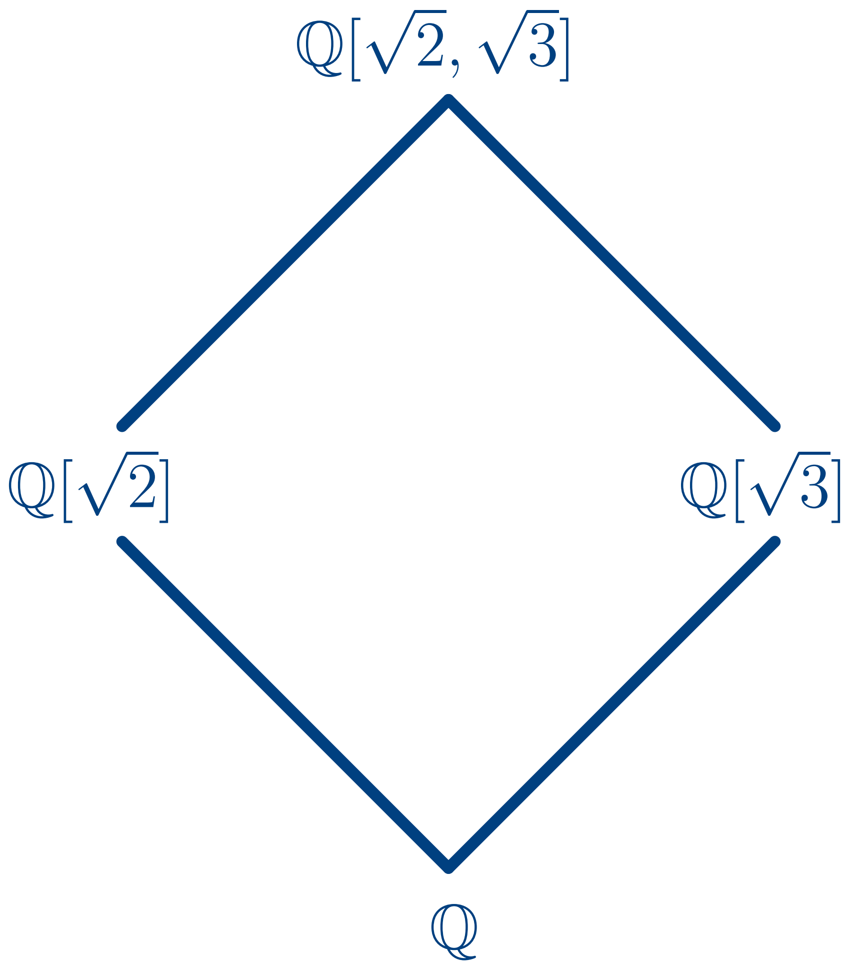

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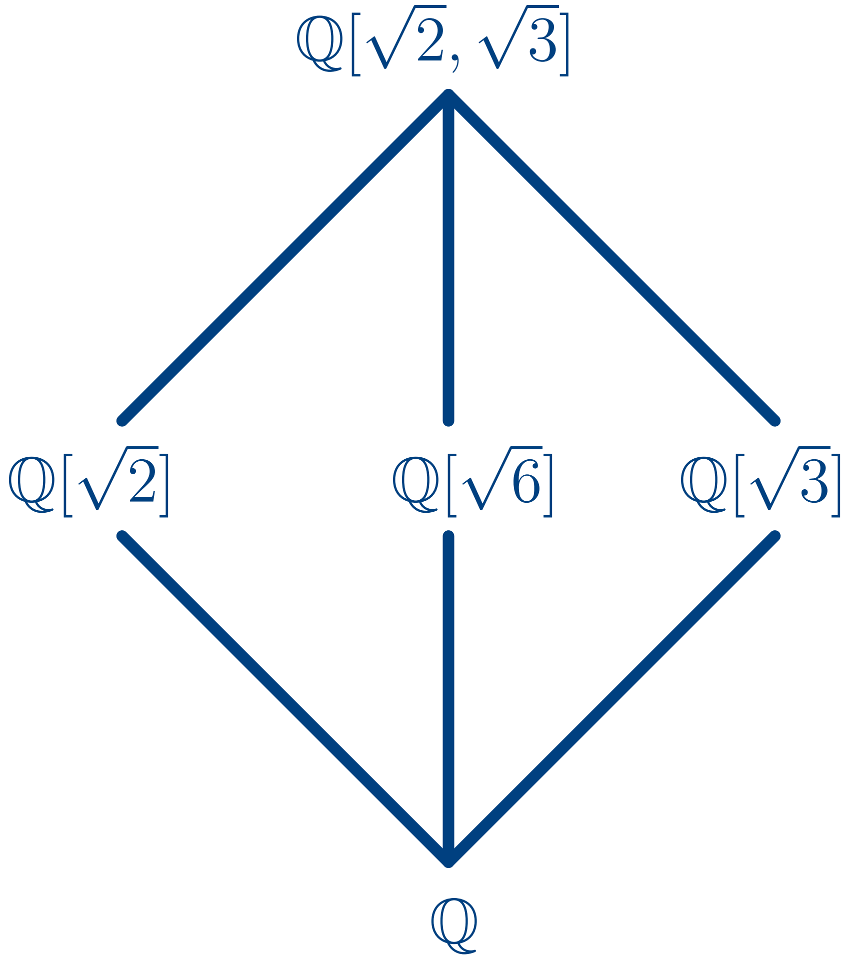

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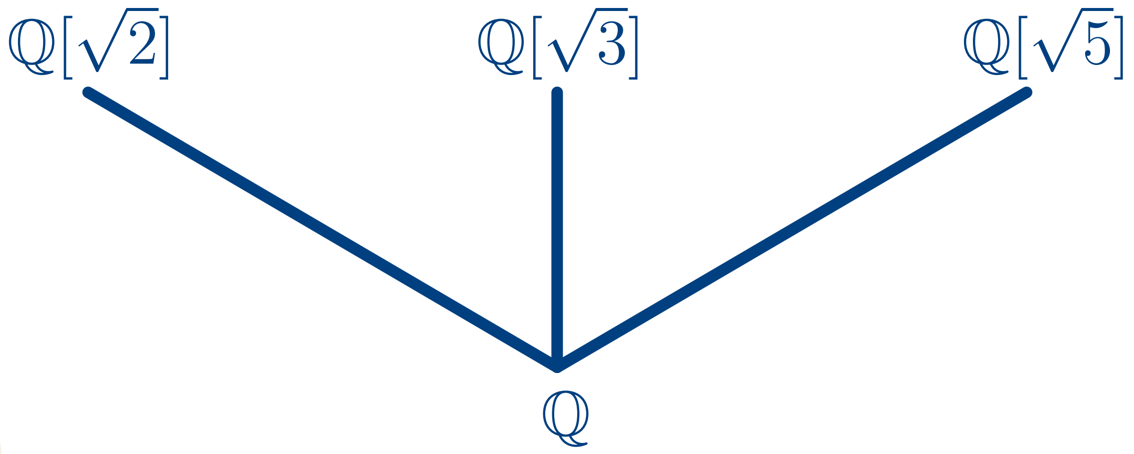

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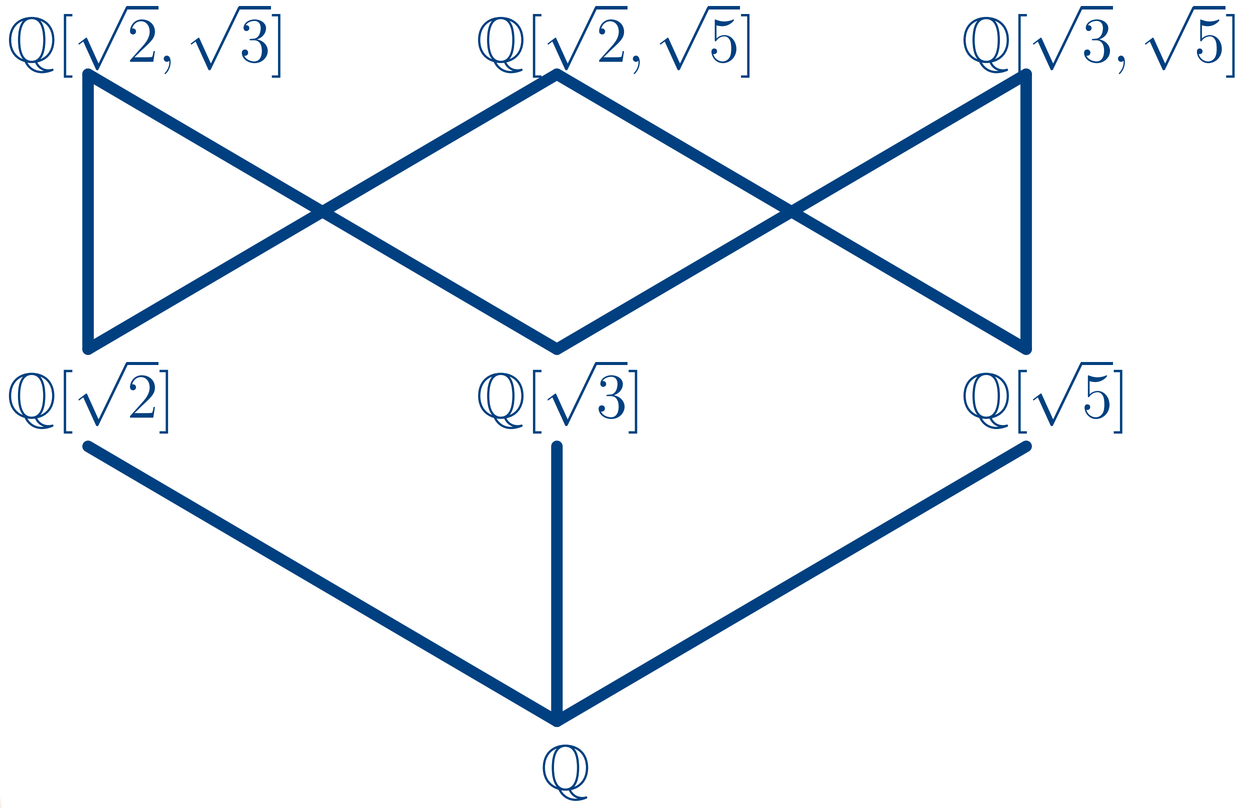

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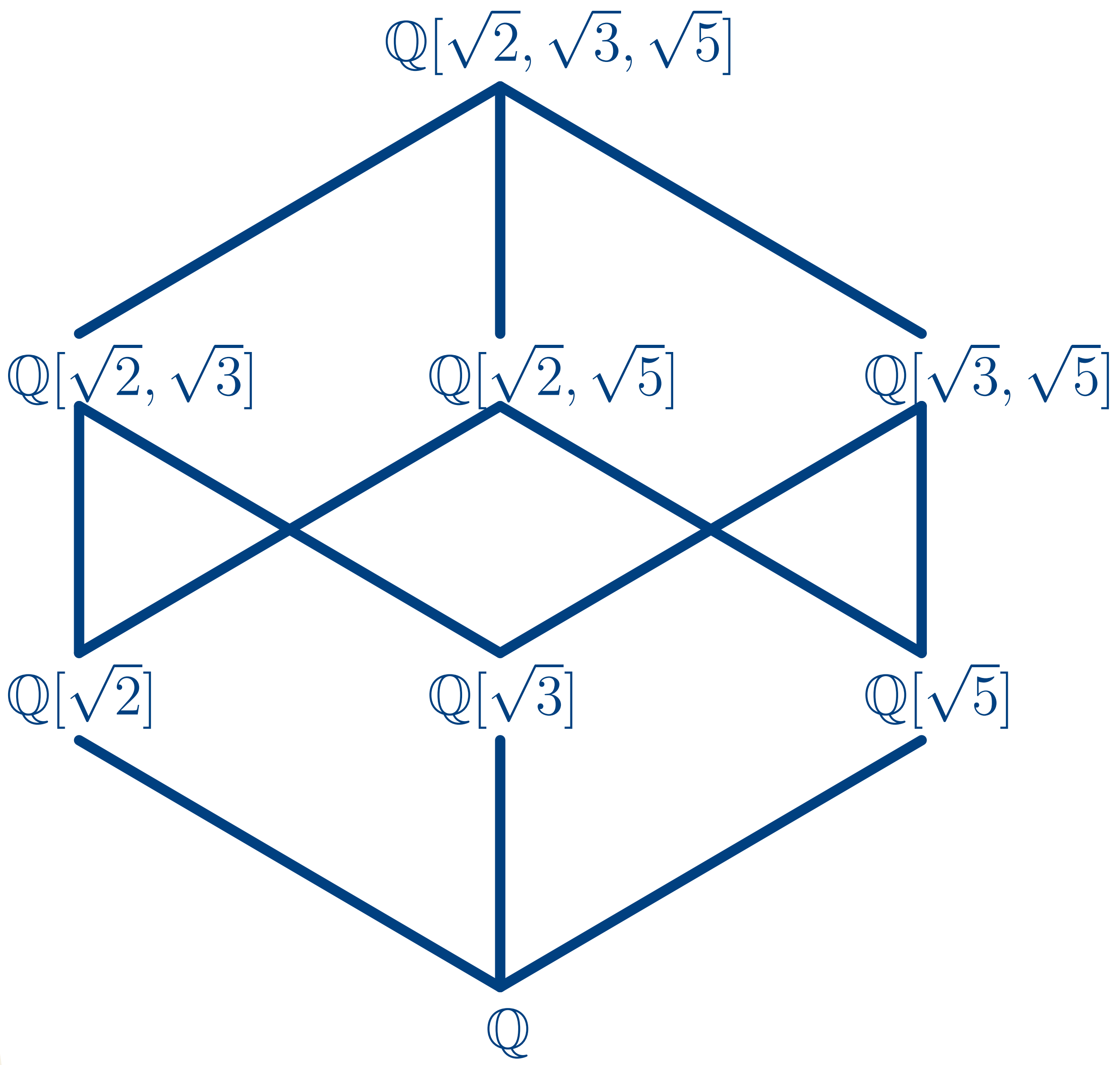

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How many total? **15**

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Note that any biquadratic field, say $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$, contains three quadratic subfields. In this case, they are $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt{3}]$, and $\mathbb{Q}[\sqrt{6}]$.

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MON	TUE	WED	THU
$2, 3, 6$	$2, 5, 10$	$2, 7, 14$	$2, 15, 30$
$5, 21, 105$	$3, 70, 210$	$3, 5, 15$	$3, 14, 42$
$7, 30, 210$	$6, 14, 21$	$6, 35, 210$	$5, 7, 35$
$10, 14, 35$	$7, 15, 105$	$10, 42, 105$	$6, 70, 105$
$15, 42, 70$	$30, 35, 42$	$21, 30, 70$	$10, 21, 210$

FRI	SAT	SUN
$2, 21, 42$	$2, 35, 70$	$2, 105, 210$
$3, 35, 105$	$3, 7, 21$	$3, 10, 30$
$5, 6, 30$	$5, 42, 210$	$5, 14, 70$
$7, 10, 70$	$6, 10, 15$	$6, 7, 42$
$14, 15, 210$	$14, 30, 105$	$15, 21, 35$

Finite Projective Geometry

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(a) there are only a finite number of points, and

(b) any two lines that are in the same plane intersect.

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Finite projective 3-space, denoted by $PG(3,q)$ contains q^3+q^2+q+1 points.

Notice that, conceivably one could take the entire space and partition it.

$$q^3+q^2+q+1=(q^2+1)(q+1)$$

Finite Projective Geometry

Every finite space has an associated *order* that tells us now many points lie on a line. If we call the order q, one can show that every line contains $q + 1$ points.

Finite projective 3-space, denoted by $PG(3, q)$ contains $q^3 + q^2 + q + 1$ points.

Notice that, conceivably one could take the entire space and partition it.

$$q^3 + q^2 + q + 1 = (q^2 + 1)(q + 1)$$

Is it possible to partition $PG(3, q)$ into $q^2 + 1$ lines?

let $q = 2$

Number of points of $PG(3,2) = 2^3 + 2^2 + 2 + 1 = 15$.

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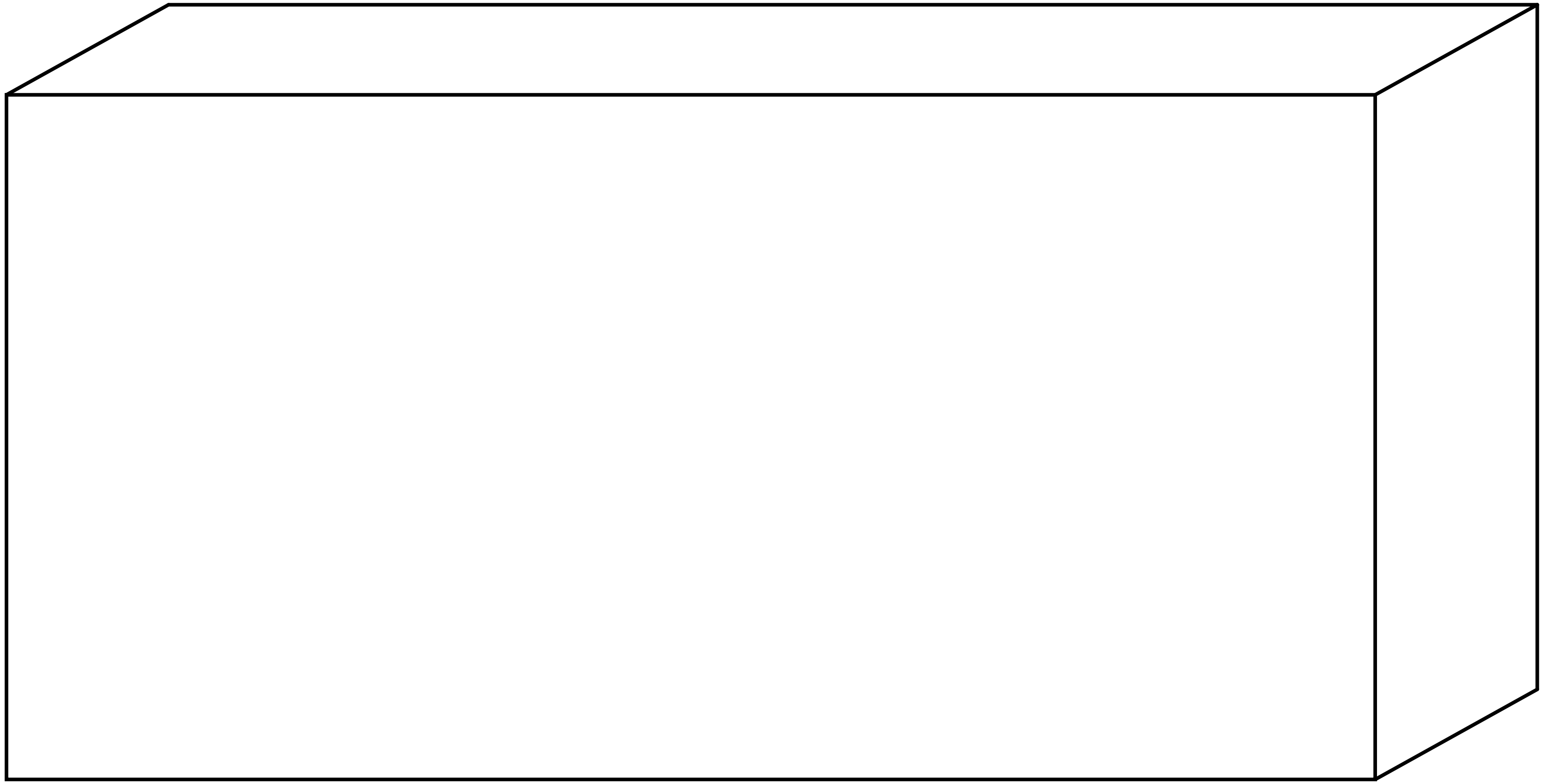
Number of lines in $PG(3,2)$ is **35**.

Partitioning 3-space



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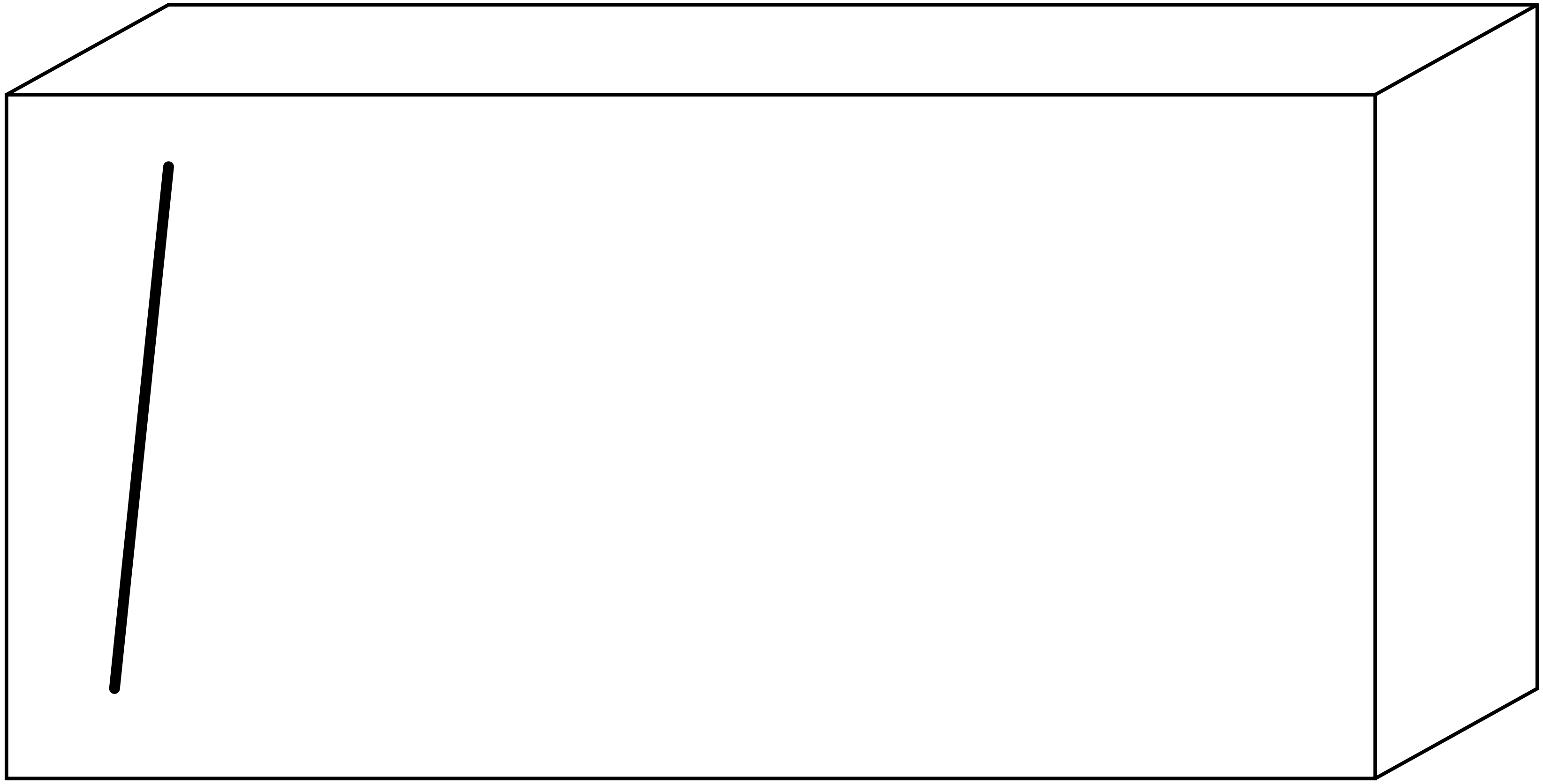

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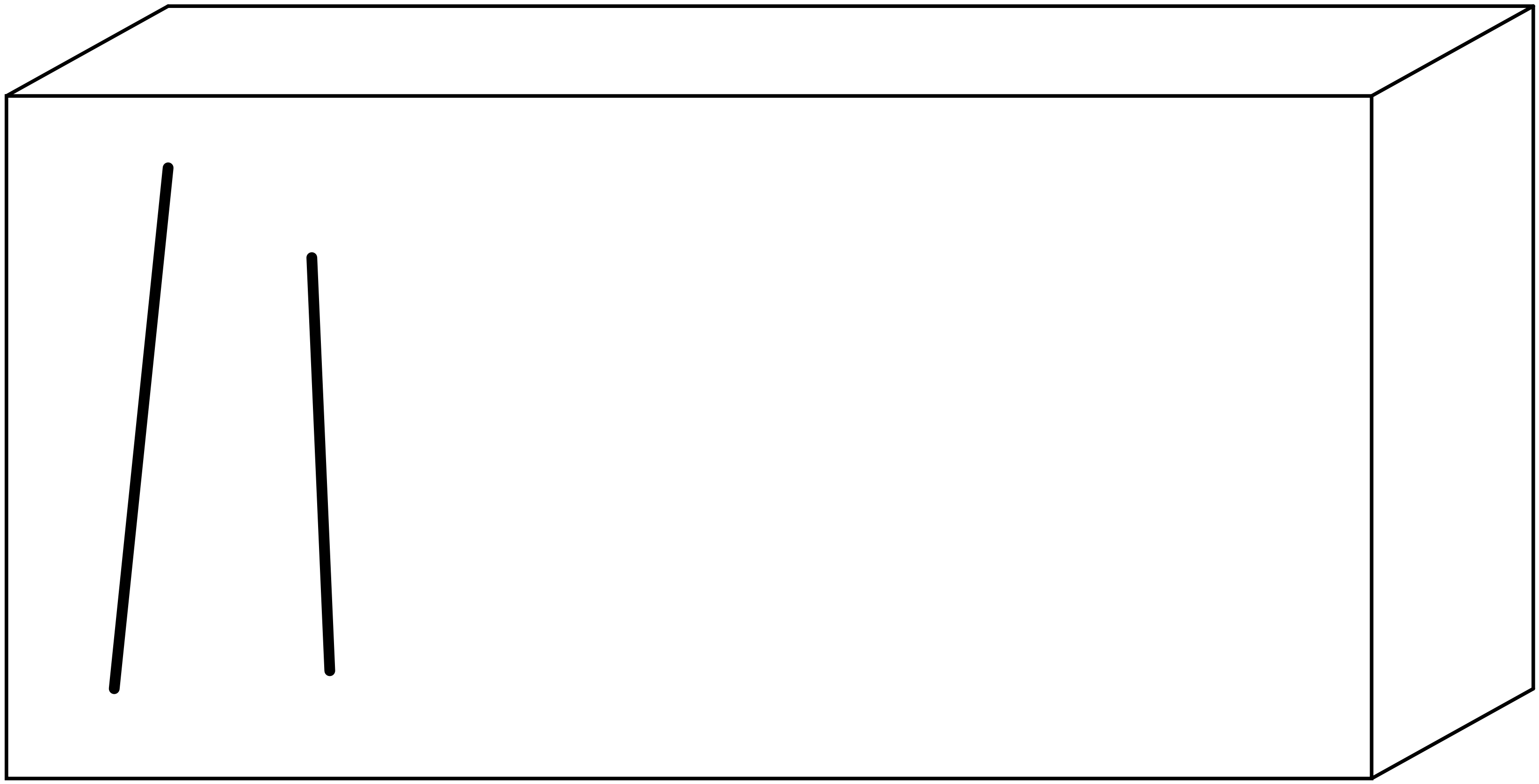

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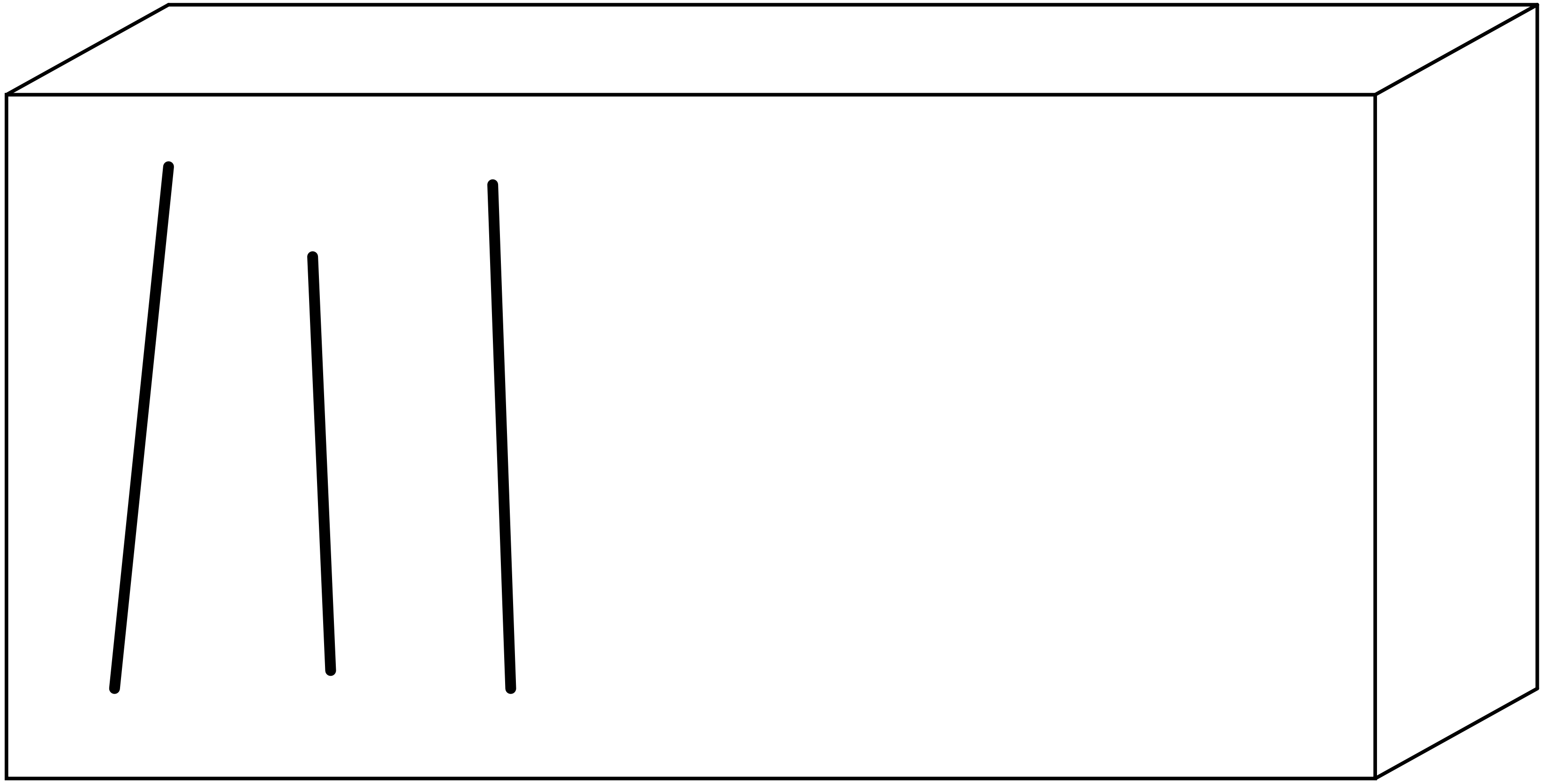

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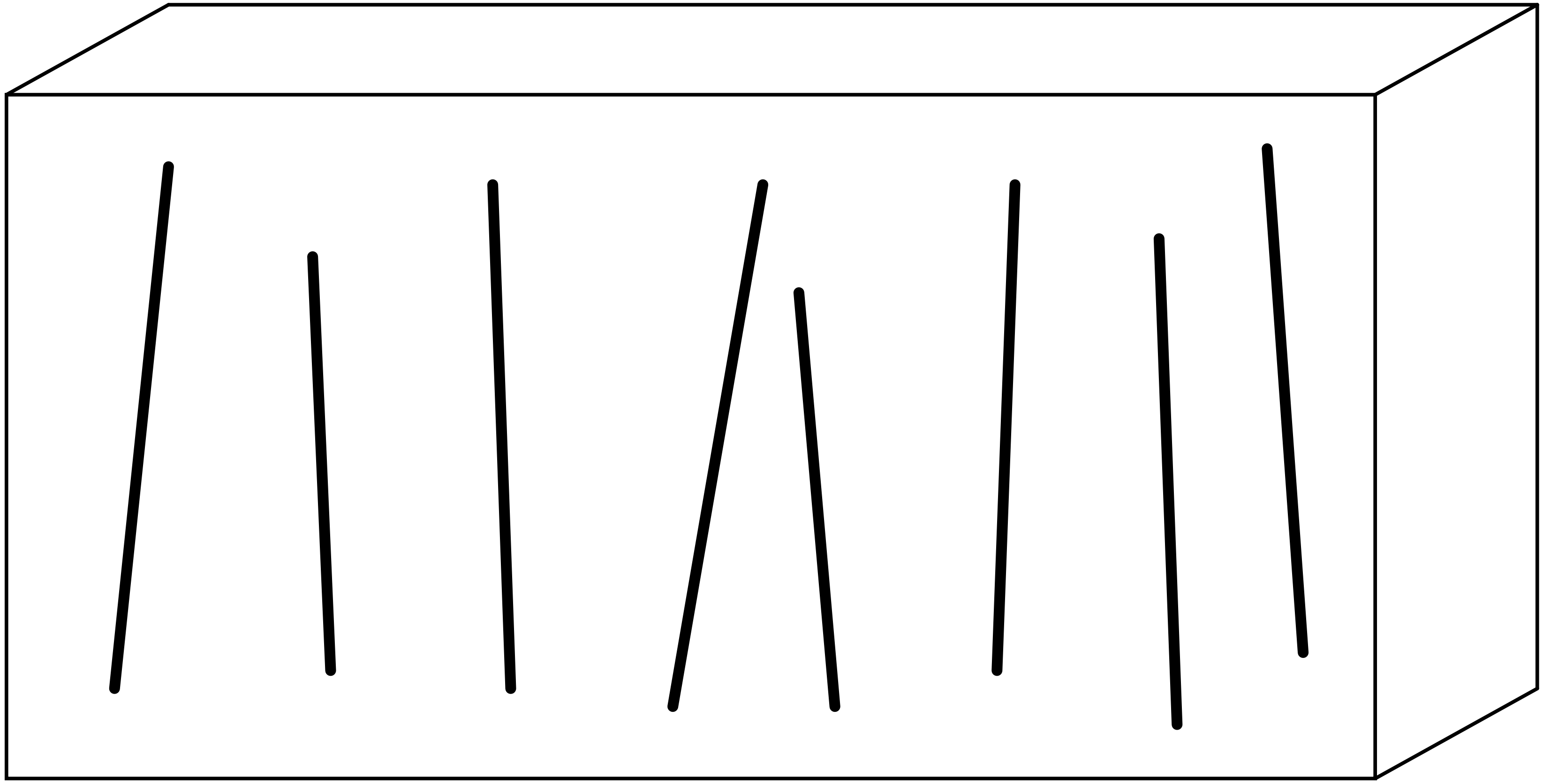

General Partitioning of 3-space

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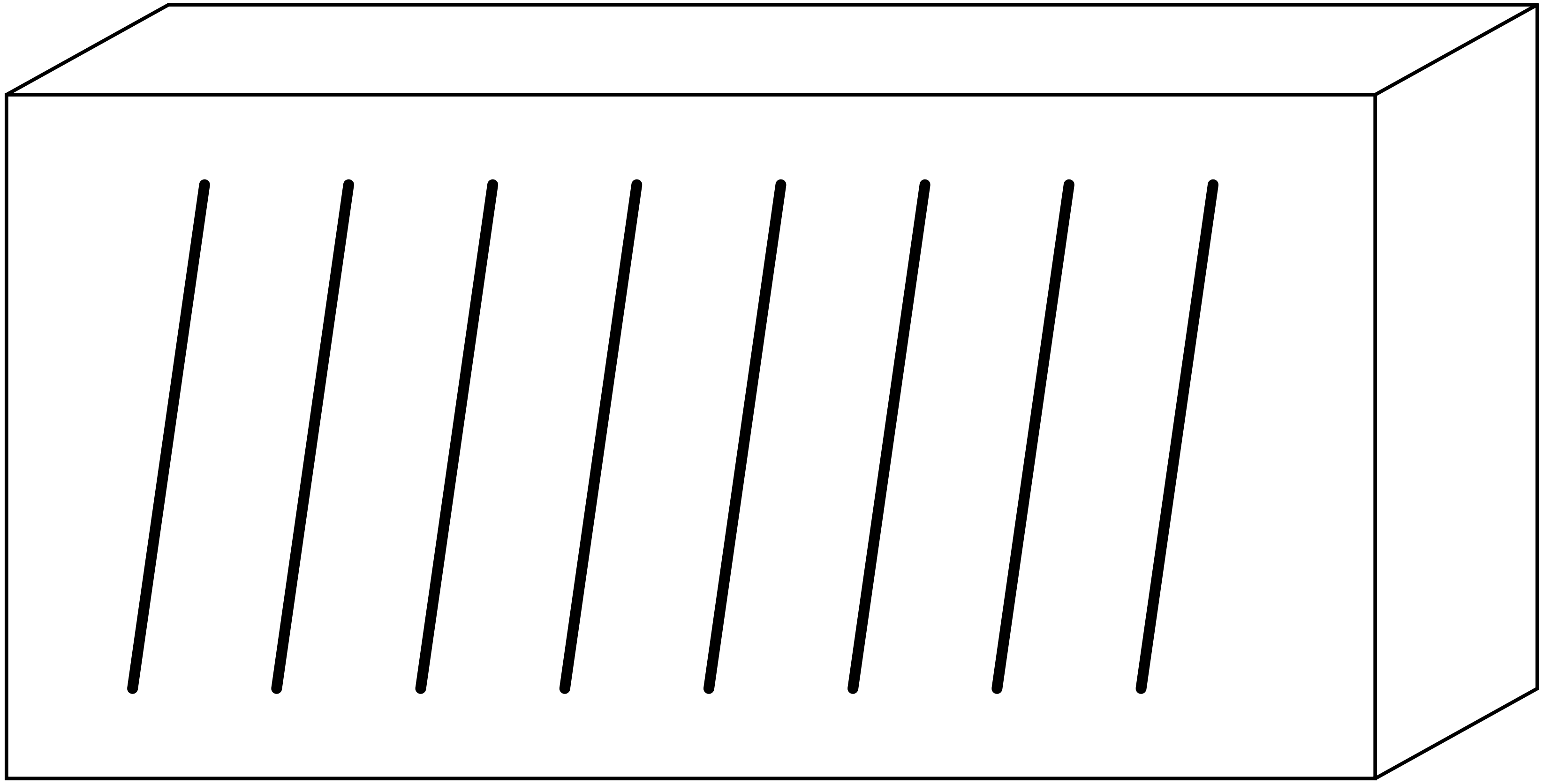
A set of $q^2 + 1$ pairwise disjoint lines would then partition the points of Σ. Such a set is called a *spread*, and spreads have a close connection of finite affine translation planes, a big are of research for some people.

Packings



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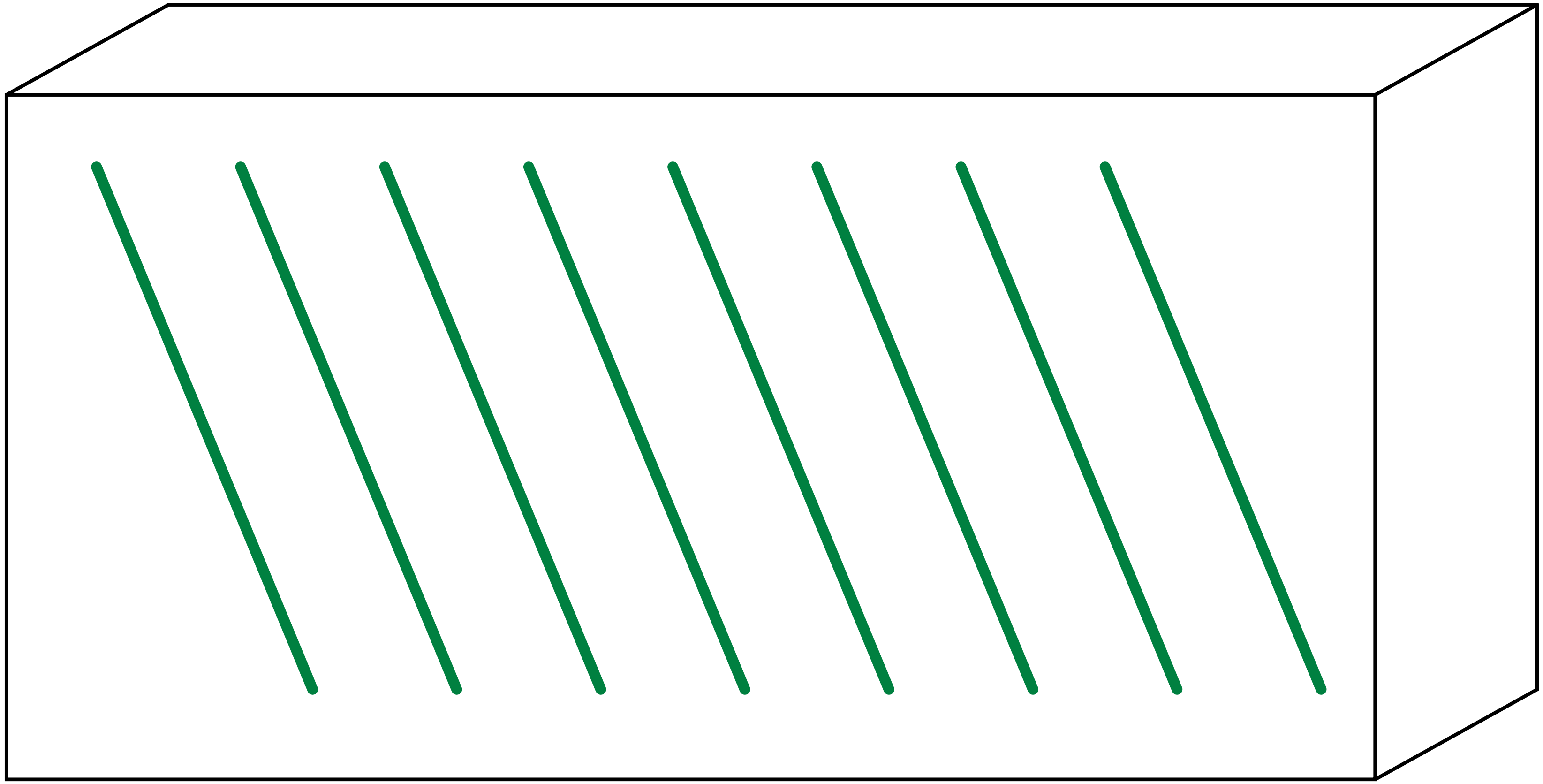

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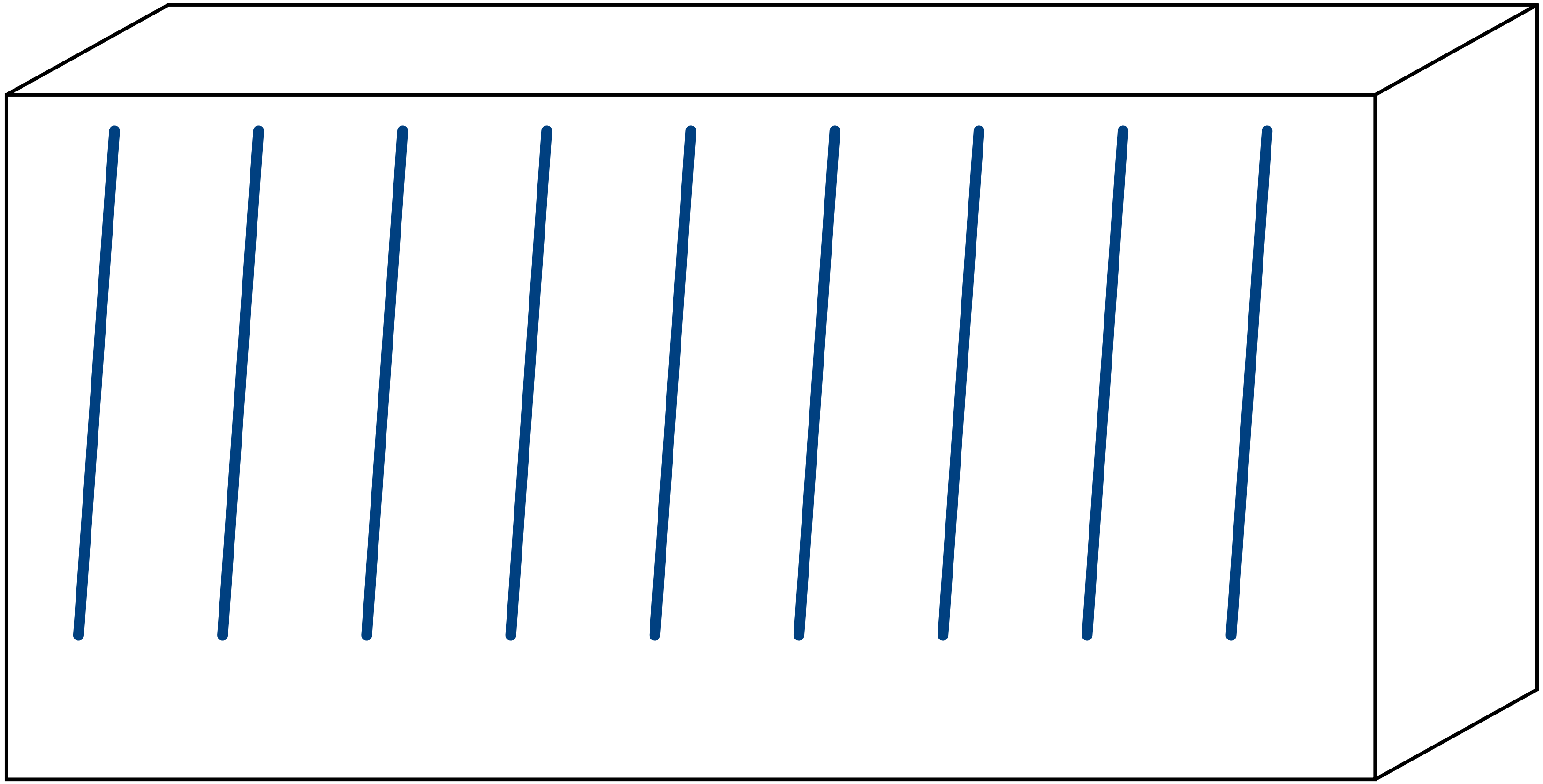

generalized solution

Spreads and packings of $PG(3, q)$, in general, give us a solution to a more generalized Kirkman Schoolgirls problem:

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If $(q^2 + 1)(q + 1)$ *schoolgirls go walking each day in* $q^2 + 1$ *rows of* $q + 1$*, they can walk for* $q^2 + q + 1$ *days so that each girl has walked in the same row as has every other girl and hence with no girl twice.*

The Hat game

Three players enter a room and a red or white hat is placed on each person's head. The color of each hat is determined by a coin toss, with the outcome of one coin toss having no effect on the others. Each person can see the other players' hats but not his own.

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The group shares a hypothetical $3 million prize if at least one player guesses correctly and no players guess incorrectly. The problem is to find a strategy whereby the group's chance of winning exceeds 50%.

history

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The problem was then popularized by an April 2001 article in the *New York Times*.

NYT article

April 10, 2001

Why Mathematicians Now Care About Their Hat Color

By SARA ROBINSON

BERKELEY, Calif., April 9 — It takes a particularly clever puzzle to stump a mind accustomed to performing mental gymnastics.

So it's no ordinary puzzle that's spreading through networks of mathematicians like a juicy bit of gossip. Known as "the hat problem" in its most popular incarnation, this seemingly simple puzzle is consuming brain cycles at universities and research labs across the country and has become a vibrant topic of discussion on the Internet.

The reason this problem is so captivating, mathematicians say, is that it is not just a recreational puzzle to be solved and put away.

Rather, it has deep and unexpected connections to coding theory, an active area of mathematical research with broad applications in telecommunications and computer science.

In their attempts to devise a complete solution to the problem, researchers are proving new theorems in coding theory that may have applications well beyond mathematical puzzles.

"This puzzle is unique since it connects to unsolved mathematical questions," said Dr. Joe Buhler, deputy director of the Mathematical Sciences Research Institute here and a hat problem enthusiast.

The hat problem goes like this:

understanding codes

A linear code is simply a vector space over the binary alphabet. If the vectors all have n entries in them, we say the code has *length* n. The dimension of the code is denoted by k.

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The last important parameter is the so-called *minimum distance* of the code.

The distance between two codewords is the number of coordinate positions where they differ. The minimum value, taken over all codewords in the code, is called the minimum distance.

Hamming code of length 3

$$\{(0,0,0),(1,1,1)\}$$

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Suppose the vector $(1,0,0)$ is received.

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KEY: For Hamming codes, every vector is either a codeword, or is distance 1 from a codeword.

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Let's redo the game now with 15 players (maybe even Schoolgirls!)

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Players assume that "they" are not in the code and answer accordingly...

to guess or not to guess...

If wearing white makes them a codeword, then the player guesses red.

If wearing red makes them a codeword, then the player guesses white.

If neither color makes them a codeword, then the player passes.

how does it work?

●
$(0, 0, 0, 0, 0, 0, 0)$

●
$(1, 0, 1, 1, 0, 0, 0)$

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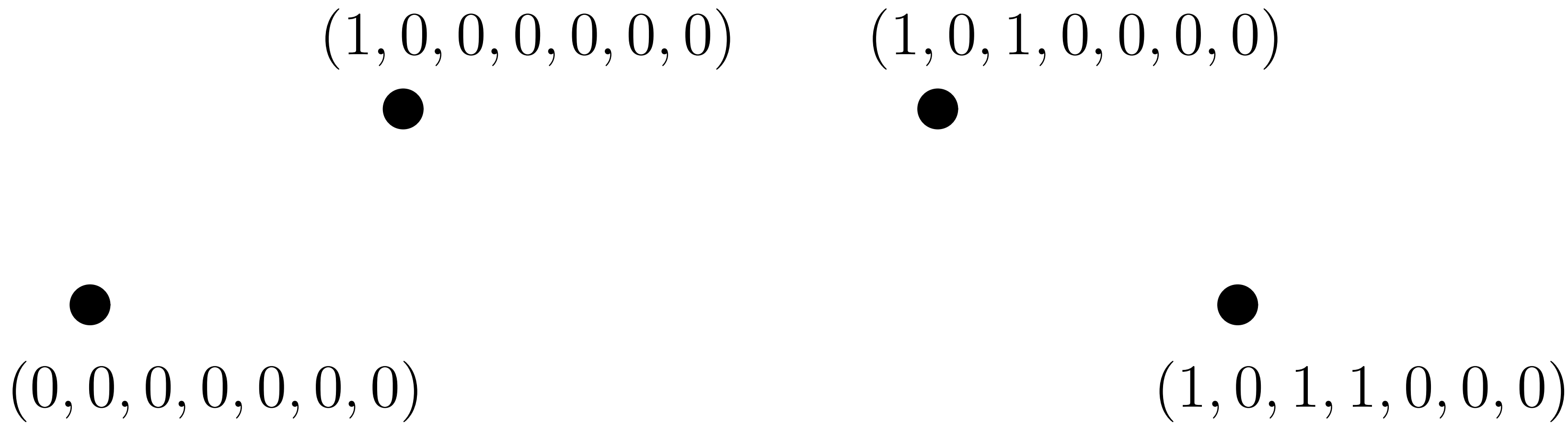

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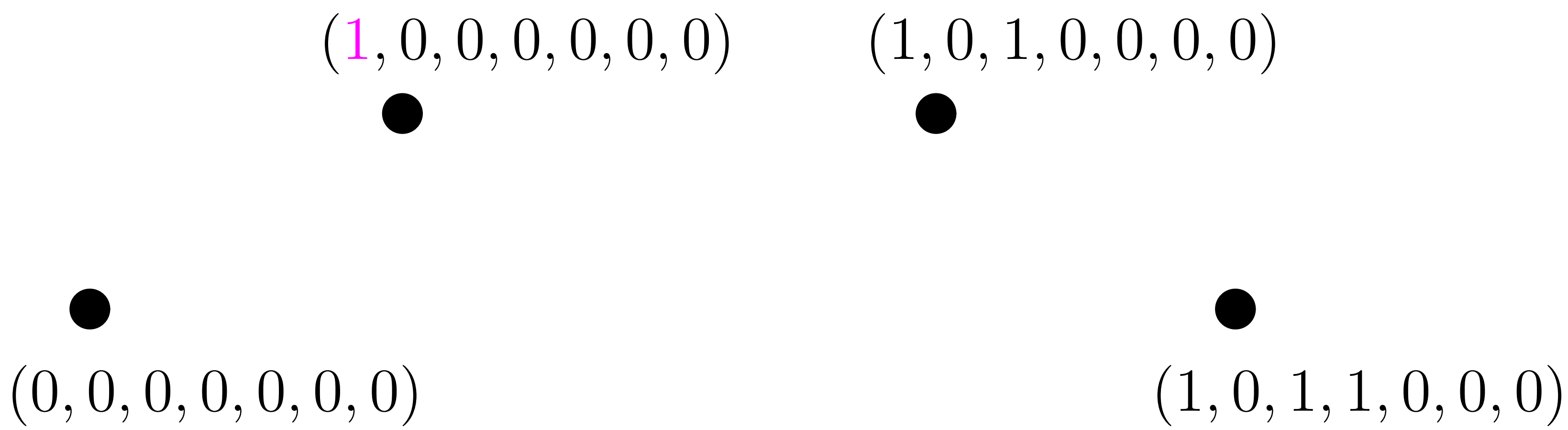

how does it work?

❖ Kirkman's Problem
❖ Block Designs
❖ Algebraic Fields
❖ Finite Geometry
❖ **Hamming Codes**
❖ Conclusions

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$$\frac{2^{11}}{2^{15}} = \frac{1}{2^4} = \frac{1}{16}.$$

In general, the probability of losing with $2^r - 1$ players is only $\frac{1}{2^r}$.

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In fact, the association will tie all of the solutions together!

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Integers: $\{1, 2, ..., 15\}$

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$$\Updownarrow$$

4-bit long bitstrings: $\{[0001], [0010], [0011], ..., [1111]\}$

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Under this ordering, the 3-tuples of integers generate the codewords in the Hamming code!

If you like what you've heard, you may be interested in reading the article

KIRKMAN'S SCHOOLGIRLS WEARING HATS AND WALKING THROUGH FIELDS OF NUMBERS

by Ezra Brown and Keith E. Mellinger

freely available on the MAA website.

- ❖ Kirkman’s Problem
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Questions?